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An unusual application of a simple geometric principle

LAURA GUGGENBUHL, *Hunter College, New York, New York.*

A simple geometric principle proves of use to plastic surgeons. It may be useful as an interesting application to discuss in a geometry class

ALWAYS ON THE LOOKOUT for practical applications, the teacher of mathematics frequently turns to such sciences as physics, chemistry, and biology. In recent years, however, mathematics has become increasingly important in the medical sciences, and today's premedical student is well advised to take courses in mathematics at least through the calculus.

On all sides one hears about the possible uses of atomic energy in the treatment of certain illnesses; most people are well aware of the numerical exactitude required in prescribing certain medicines; and even the simplest concepts of geometric form and position are necessary for certain reconstructive surgical processes.

The following paragraphs describe the mathematical basis of an operation known as "The *Z* Plastic Procedure." A matter of common knowledge in the annals of plastic and reconstructive surgery,[1] it is based upon the simplest possible geometric principle. Since the writer has seen no reference to the *Z* Plastic Procedure in mathematical literature, this note may be of more than passing interest.

Though there will be no attempt to discuss the matter from the medical or surgical point of view, a few introductory remarks are necessary. To understand the circumstances under which this operation is used, one must think of a certain type of disability. For example, as the aftermath of a burn, a constricting scar in the armpit may make it difficult or impossible to raise the arm above the shoulder. A severe neck burn, upon healing, has been known to leave the head in a rigid position inclined toward the chest. Other types of injuries or even congenital malformations (particularly those in webbed sections of the body's surface) have also caused similar limitations of function. To reduce considerations to the simplest possible terms, let us think of a lineal scar of fixed length which has contracted the skin in such a way as to interfere with the normal action of adjoining muscles.

Although the above example is based upon the assumption that the vertex angle of the skin flap is 60 degrees, physical conditions sometimes indicate the choice of an angle of different size. Whether or not the angle is 60 degrees, the mathematician will see the following abstractions in an ideal case.

Using the conventional notation of plane geometry and trigonometry, we may say that in triangle ACB

$$c^2=a^2+b^2-2ab\cos C.$$

Let us designate the constant length of

[1] The year 1956 marked the one hundredth anniversary of the date of the first well-known article on this subject. The article "Blépharoplastie" by M. Charles Pierre Denonvilliers—only twenty lines long, and without illustrations or diagrams—appeared in 1856 in Vol. 7, p. 243, of the *Bulletin de la Société de Chirurgie de Paris.*

the scar CD by the letter l. Then we may say further that

$$AC=CD=DB=l.$$

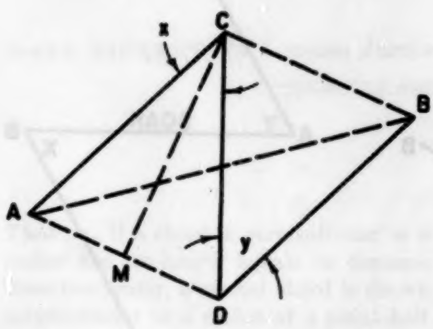


Figure 1

Let CM represent the perpendicular bisector of side AD in triangle ACD. Then we may say

$$CB=AD=2AM=2l\sin\frac{x}{2}\cdot$$

Substituting $2l\sin x/2$ for side a, l for b, and

$$x+\frac{180-x}{2}$$

for angle C in the Law of Cosines above, and reducing, gives

$$c^2=4l^2\sin^2\frac{x}{2}+l^2+4l^2\sin^2\frac{x}{2}\cdot$$

Substituting

$$\frac{1-\cos x}{2}$$

for $\sin^2 x/2$ and simplifying, gives

$$c=l\sqrt{5-4\cos x}.$$

If we designate the amount of relaxation (the difference between the longer and shorter diagonals of the parallelogram $ACBD$) by y, we have

$$y=l(\sqrt{5-4\cos x}-1).$$

The latter statement may be regarded as a formula which expresses the amount of relaxation as a function of the constant scar length and the angle of the skin flap. If the angle x is taken as 60 degrees, the amount of relaxation is obviously $l(\sqrt{3}-1)$ or $.73l$.

The length l and the number of degrees in the angle x are determined by the physical realities of the individual case. It is interesting to note that in some cases l has been taken as small as 5 mm., and that, on the other hand, it has also been taken as large as 20 cm.

The above account represents only the most elementary application of "The *Z* Plastic Procedure." No doubt the surgeon considers this part of the operation as the simplest part of the entire problem. Nevertheless, one may stand in humble and silent admiration at the thought that a geometric principle of such elegant simplicity is the basis of a surgical procedure of such miraculous proportions.

In Figure 2, the line CD represents such a scar.[2] The surgeon outlines the figure $ACDB$, where CA and DB are both taken equal to CD, and inclined at an angle of about 60 degrees to CD. In actual practice, the angles ACD and CDB are rounded off because of medical considerations. The surgeon then cuts the skin along this pattern to produce two free triangular skin flaps, ACD and CDB. The flaps are then interchanged. The flap CDB is placed with its edge CD adjacent to CA, with the corner Y at the vertex A, Figure 2. The flap ACD is placed with the edge CD next to BD, and the corner X at the vertex B. The flaps are then sewed into place in this new position, and the scar now lies

[2] Figures 2 and 3 are after John Staige Davis, *Annals of Surgery*, Vol. XCIV (July 1931).

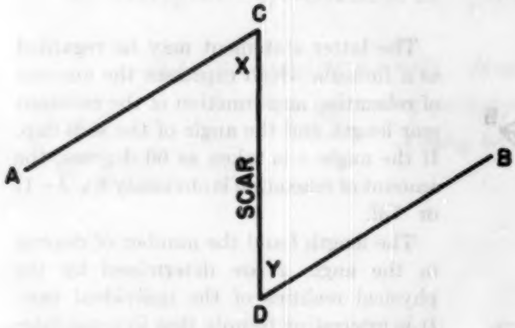


Figure 2

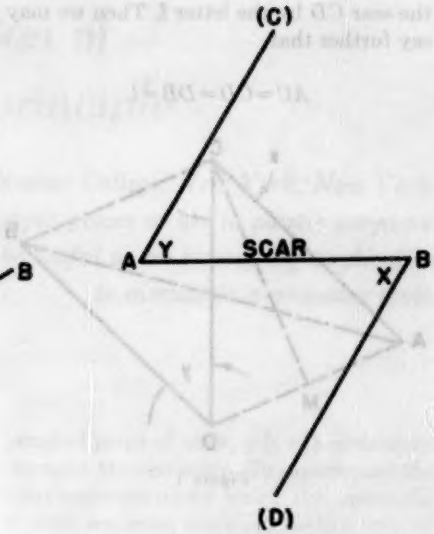


Figure 3

along the perpendicular bisector of the original line of the scar.

Thus if the original scar pull had been in the vertical direction, it has now been transformed into the horizontal direction.[3] The scar remains, but its crippling effects have been rendered less potent. At the same time, the longer diagonal AB of our parallelogram $ACBD$, Figure 1, has been placed in a more useful position (C and D of Figure 3), to provide what is called a relaxation of skin tension. It is also significant to note that the surgeon may predict in advance, by a simple mathematical computation, the amount of relaxation which this procedure will yield.

[3] If the reader wants to make a model, it is recommended that a piece of chamois cloth rather than a piece of paper be used.

Have you read?

MARTIN, E. W., "The Challenge of the Electronic Computer," *Business Horizons*, supplement to *Indiana Business Review*, January, 1957, pp. 22–28.

All of us are curious as to the future limitations of the computing machines. We also wonder how far they can extend the limit of human judgment. This article not only gives pertinent information on the above, but adds to it information of special interest to all of us in mathematics. It is amazing to realize that all the current and stored information is a number language; that directions are issued in this number language; that order is the key to success; and that there is an average of twenty million steps between errors—and even these are then detected and corrected by the machine.

But the human element becomes more important than ever because someone must determine how to solve the problem in every minute detail. If the machine is fed incorrect procedures, it becomes the fastest means of getting wrong answers.

Computing machines have resulted in a rapid advancement in the use made of matrix algebra, linear programming, probability, and others. You will also be interested in the comments on the potential value of the computing machine in providing up-to-the-minute information.

Length of chords and their distances from the center

HALE PICKETT, *El Camino Junior College, El Camino College, California.*
Stimulating supplementary material for classroom use is always welcomed by geometry teachers.

Theorem: If a chord is perpendicular to a radius and its length equals its distance from the center, a second chord is drawn perpendicular to a radius at a point half the distance of the first chord from the center: then the second chord equals twice the length of the first chord.

Figure 1

Given:

EF perpendicular to OA $\quad EF=OH$

CD perpendicular to OK $\quad OL=1/2\ OH$

To prove:

$$CD=2EF$$

Proof: Draw OF and OD radii of the circle.

Let $HF=d$. Then $OH=2d$ and $OL=d$.

$$(OH)^2+(HF)^2=R^2$$
$$4d^2+d^2=R^2$$
$$(OL)^2+(LD)^2=R^2$$
$$d^2+(LD)^2=R^2$$

Then

$$4d^2+d^2=d^2+(LD)^2.$$
$$4d^2=(LD)^2$$

Then

$$LD=2d,\quad LD=EF,\text{ and } CD=2EF.$$

Note $EF=OH=LD$.

This theorem was developed as a solution to the exercise, which occurs in many plane geometry textbooks: ***If we double a chord, do we halve its distance from the center?***

The answer is *yes*, if we select the proper chord; however in general, the answer is *no*. Almost all students miss this exercise and often teachers of geometry fail to give the complete solution.

This solution suggests another problem, namely: ***Given a circle, construct a chord perpendicular to the radius whose length equals its distance from the center.*** Refer to Figure 1.

$$(OH)^2+(HF)^2=R^2$$
$$4d^2+d^2=R^2$$
$$5d^2=R^2$$

Then

$$d=\frac{R}{\sqrt{5}}.$$

Point H is located $2d$ distance from the center of the circle. Since this construction involves $\sqrt{5}$, we must first construct a line $\sqrt{5}$ units in length, and then proceed with the final construction. Let R, the radius of the circle, be the unit of measure.

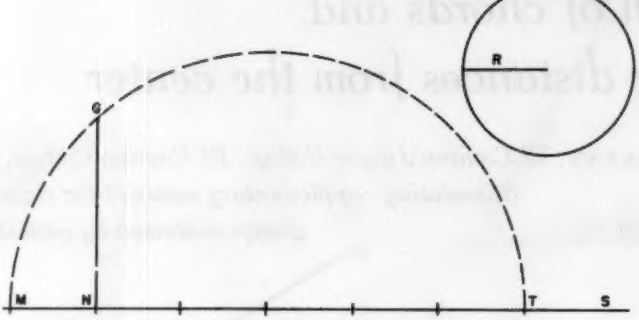


Figure 2

On the working line MS (Figure 2), lay off $MN = R$; $NT = 5R$.

$$\frac{MN}{NG} = \frac{NG}{NT}$$

$R = 1$, the new unit of measure. Substituting

$$\frac{1}{NG} = \frac{NG}{5}.$$

$$(NG)^2 = 5, \quad \text{then} \quad NG = \sqrt{5}.$$

The second construction involving $\sqrt{5}$ is

$$d = \frac{R}{\sqrt{5}}.$$

Substituting $R = 1$,

$$d = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}.$$

In Figure 2 the line $NG = \sqrt{5}$, when R, the radius of the circle, becomes unity.

The problem now is to divide NG, which equals $\sqrt{5}$, into five equal parts.

Construction:

Lay off line NG (Figure 3), which is to be divided into five equal parts. Through point N lay off line NZ, any length and at any angle with NG.

On line NZ with any convenient radius, lay off five equal line segments. Then through points J, V, P, and Y construct lines parallel to XG.

$$IG = 1/5\, NG, \text{ or } IG = d, \text{ and } 2d = UG.$$

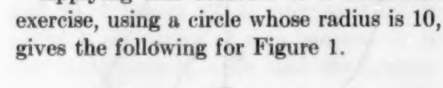
Therefore at a distance UG from the center of a circle construct a chord perpendicular to the radius and in accordance with the theorem and previous constructions. The chord will be equal to its distance from the center.

Applying this theorem to a numerical exercise, using a circle whose radius is 10, gives the following for Figure 1.

$$d = \frac{10}{\sqrt{5}} = \frac{10\sqrt{5}}{5} = 2\sqrt{5} = 4.472$$

Therefore if $OA = 10$; $OH = 8.944$; $EF = 8.944$; $CD = 17.888$.

Figure 3



The ABC's of geometry

JOHN D. WISEMAN, JR., *Tenafly High School, Tenafly, New Jersey.*

Here is a statement of some difficult learning spots in geometry and suggested means of overcoming some of these difficulties.

MANY ARTICLES in THE MATHEMATICS TEACHER propose that high school geometry adopt the logical pattern of *if-then* reasoning. This form is characteristic of mathematics itself.[1] Sitomer advocated that the *if-then* form should be used not only for the theorems of geometry, but also for the assumptions and definitions.[2] I have followed these suggestions in my geometry classes.

My students readily accept the idea that statements about geometric material should be put into the *if-then* form. Only one difficulty has appeared—changing definitions. Definitions usually are stated this way:

> Congruent triangles are triangles whose corresponding sides and angles are equal.

My students have been instructed to change the definition into the *if-then* form in this manner: (1) write "if" in front of the definition, (2) insert "then" in place of the verb "is" (or "are"), (3) make any minor changes in wording necessary for good grammar.

It should be noted that the verb "is" will always appear since a definition is a statement of equivalence between the word defined and the defining phrase. This idea of equivalence, the *if-and-only-if* relationship, has not been presented to the students. They have been assured only that the change can always be made in the foregoing manner. Thus the definition used as an example becomes in *if-then* form:

> If two triangles are congruent, then they are triangles whose corresponding sides and angles are equal.

There is one further complication. Frequently it is necessary to use a definition "turned around." In the course of a proof, it may have been established that two triangles have their corresponding sides and angles equal. Then the conclusion is that the two triangles are congruent. This means that the definition must be used with the *if*-part and the *then*-part interchanged. Thus:

> If two triangles have their corresponding sides and angles equal, then the triangles are congruent.

My students have been given practice in changing each of their definitions into an *if-then* statement, and then interchanging the parts to get the second *if-then* statement. This seems to be a sound classroom procedure because it brings out intuitively the meaning of equivalence.

It is not too difficult for students to put the statements of geometry into *if-then* form. What are the advantages?

The first advantage is that students can be introduced to a procedure in proof that is used throughout mathematics. The familiar procedure in proving a theorem is to draw a triangle *ABC*, for example, and to state the proof in terms of this specific triangle. If the students are told anything at all about how triangle *ABC* is obtained, it is only that it is an arbitrarily selected triangle. But this is a wonderful

[1] Roland R. Smith, "Three Major Difficulties in the Learning of Demonstrative Geometry," THE MATHEMATICS TEACHER, XXXIII (April 1940), 118.

[2] Harry Sitomer, "If-then in Plane Geometry," THE MATHEMATICS TEACHER, XXXI (November 1938), 327.

opportunity to present to the students a fundamental principle of logic that is used in mathematics. It is called the ***principle of specification.***[3] This principle is written symbolically:

$$(x): f_x \supset f_z$$

This formula asserts that what is true for all values of "x" is true for a specific value of "x," namely "z."

I have presented this principle to my geometry classes in the following form (the example is the *pons assinorum*):

If any triangle has two equal sides,
If in $\triangle ABC$ $\quad AB=AC$,

then the two angles opposite are equal.
then $\angle B=\angle C$.

This pattern presents geometric reasoning as a pair of *if-then* statements, the first a general statement, the second a specific statement. The *if-then* relation between the general statement and the specific statement is not expressed. But it is implied by the position of the specific statement directly underneath the general statement. Thus students in my geometry classes know what it means to *specify*, that is, to take an *if-then* statement and translate it into words which talk about the figure they have drawn.

A second advantage is that the *if-then* form makes a direct attack upon two of the three principal difficulties in the learning of demonstrative geometry. Smith mentions these difficulties:

1. Although pupils can make a deduction when simple everyday situations without any complications are given, they cannot necessarily do so when there are complications even in simple situations.

2. Pupils need to learn that a conclusion can be drawn only when all the conditions are fulfilled.

3. They must learn to analyze a statement to find out what the conditions are that must be fulfilled. and they must be able to see whether all the conditions have been fulfilled.

4. They must learn that when the conditions are fulfilled, the only conclusion that can be drawn is the one stated in the general statement.[4]

It is evident that the form used with my students is designed to meet the difficulties in 2, 3, and 4. The *if*-part of the specific statement is written directly underneath the *if*-part of the general statement and often is a word-for-word translation of it. The same applies to the *then*-part of the specific statement. It is written directly underneath the *then*-part of the general statement and often is a word-for-word translation of it. Thus, students readily see that the two *if-then* statements have the same meaning and that the *if*-parts and the *then*-parts correspond to each other. I think that this form eases students past these two difficulties rather well.

A third advantage is the ease with which a theorem can be generalized. A recent article in THE MATHEMATICS TEACHER by Rosskopf and Exner has pointed out that generalization is not usually made at the end of a theorem in geometry though it is logically required.[5] Only if triangle ABC, for example, is obtained at the beginning of the theorem by the principle of specification is the principle of generalization valid at the end of the proof. All this is not too formidable for high school students. I have presented the principle of generalization in the following manner. In the illustration, only the last two lines of the proof are written.

If in $\triangle ABC$ $\quad AB=AC$,
If any triangle has two sides equal,

then $\angle B=\angle C$.
then the two angles opposite are equal.

All the students have to do to generalize is to restate the sentence that refers to the figure in general terms. The students also

[3] John C. Cooley, *A Primer of Formal Logic* (New York: Macmillan, 1947), p. 94.

[4] Smith, *op. cit.*, p. 161.

[5] Myron F. Rosskopf and Robert M. Exner, "Some Concepts of Logic and their Application in Elementary Mathematics," THE MATHEMATICS TEACHER, XLVIII (May 1955), 295.

understand the meaning of the generalization. That is, since they have proved that a representative isosceles triangle, ABC, has its base angles equal, then any isosceles triangle has its base angles equal.

A fourth advantage is that the *if-then* form emphasizes the use of the student's notebook. A geometry notebook which contains lists of definitions, assumptions, and theorems together with lists of ways-and-means to prove angles equal, line segments equal, etc., can be the student's "best friend" in proving a theorem. Each of the general statements used in this pattern of proof appears in the student's notebook, and once the correct statement is selected, it need only be copied into the proof. The student then must translate this general statement into the specific *if-then* statement about the figure used in the proof. Thus the pattern of proof in geometry becomes pairs of *if-then* statements, with the first statement a general statement taken from the notebook. The second statement is a specific *if-then* statement translated from the preceding general statement. The only exception to this pattern is the generalization at the end of the theorem with the reversal of the specific and general statements.

This article presents an aspect of proof used with high school students. I think that this form is superior to the usual step-and-reason form of proof. It eases students past the major difficulties in learning demonstrative geometry, and it presents a clearer picture of mathematical reasoning.

What's new?

BOOKS

Secondary

Mathematics for Everyday Affairs (2nd ed.), Virgil S. Mallory, Syracuse, New York, L. W. Singer Company, 1956. Cloth, ix+493 pp., $3.60.

Trigonometry for Secondary Schools (2nd ed.), Charles H. Butler and F. Lynwood Wren, Boston, D. C. Heath and Company, 1957. Cloth, vii+360 pp., $2.96.

College

Advanced Real Calculus, Kenneth S. Miller, New York, Harper and Brothers, 1957. Cloth, viii+185 pp., $5.00.

Finite Mathematics, John G. Kemeny, J. Laurie Snell, Gerald L. Thompson, Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1957. Cloth, xi+372 pp., $5.00.

Introduction to Statistical Analysis (2nd ed.), Wilfrid J. Dixon and Frank J. Massey, Jr., New York, McGraw-Hill Book Company, 1957. Cloth, xiii+488 pp., $6.00.

The Basic Concepts of Mathematics, Karl Menger, Chicago, Illinois Institute of Technology Bookstore, 1957. Paper, vi+93 pp., $1.40.

The Mathematics of Investment, Roger Osborn, New York, Harper and Brothers, 1957. Cloth viii+279 pp., $4.25.

Miscellaneous

Diophantische Approximationen (reprinted with corrections), Hermann Minkowski, New York, Chelsea Publishing Company, 1957. Cloth, viii+235 pp., $4.50.

Go, the National Game of Japan (reprint), Arthur Smith, Rutland, Vermont, Charles E. Tuttle Company, 1956. Paper, xv+220 pp., $1.75.

History of Analytic Geometry, Carl B. Boyer, New York, Scripta Mathematica, 1956. Cloth, ix+291 pp., $6.00.

Printers' Arithmetic, F. C. Avis, New York, Philosophical Library, Inc., 1956. Cloth, 148 pp., $4.75.

Statistics of State School Systems: Organization, Staff, Pupils, and Finances 1953–54, chapter prepared by Samuel Schloss, Carol Joy Hobson, Emery M. Foster, Washington, D. C., U. S. Department of Health, Education and Welfare. Paper, ix+141 pp., $0.55.

The Enjoyment of Mathematics, Hans Rademacher and Otto Toeplitz, translated by Herbert Zuckerman, Princeton, N. J., Princeton University Press, 1957. Cloth, 204 pp., $4.50.

More new exercises in plane geometry

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The stock of geometrical exercises of the new type introduced in the February 1957 *issue of* THE MATHEMATICS TEACHER *is here increased so that a wide range of suggested problems is opened to the teacher of geometry. This paper, the last in a series of five devoted to the cutting of squares, also supplements similar material in* The Mathematics Student Journal *for April* 1957.

THE PRESENT PAPER is the last in a series of five papers that have appeared in THE MATHEMATICS TEACHER over the past year.[1] These papers have dealt with the theme of cutting squares; more precisely, with the theme of transforming squares into other specified geometric figures, and conversely. The first three papers—those of May, October, and December 1956—treat eight standard problems, and seek with their help to introduce both a point of view and some general methods that are effective respecting all such transformation problems. The fourth paper, that of February 1957, states and solves seven exercises which are of the same type as the initial eight standard problems, but for the most part easier. Actually, the seven exercises of our February paper are designed to make essentially no *technical* demands on our first three papers. The present fifth paper continues the February paper with four more exercises of the same sort (designated "Exercises 8," "Exercise 9," etc., in continuation of the numbering system used in the February paper). These additional exercises are also essentially independent of the previous four papers in the sense that their solutions can for the most part be understood without a reading of earlier papers. Of course, the background provided by the earlier papers increases the significance of these exercises.

As in the previous papers, so here the material is based in part on portions of the book *The Wonders of the Square* by B. Kordiemski and N. Rusalev.

EXERCISE 8 AND ITS SOLUTION

Our solution of Problem V in THE MATHEMATICS TEACHER for October 1956 showed how to transform a single equilateral triangle into a square. Exercise 8 below concerns transforming two equilateral triangles into a square. There are two obvious directions in which to seek a solution of this exercise. One direction yields a solution with six parts, the other with five. We will discuss both solutions.

Exercise 8: *Given two congruent equilateral triangles, to transform them into a single square.*

Solution 1 *of Exercise* 8: Suppose one of our two given congruent equilateral triangles is cut along an altitude, as shown in

[1] The first four papers of the series are: "A Problem on the Cutting of Squares," XLIX (May 1956), 332–346; "More on the Cutting of Squares," XLIX (October 1956), 442–454; "Still More on the Cutting of Squares," XLIX (December 1956), 585–596; and "New Exercises in Plane Geometry," L (February 1957), 125–135.

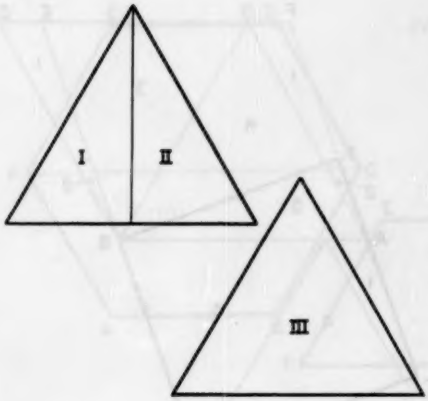


Figure 1

Figure 1. The resulting two parts, I and II, together with the second triangle (Part III), can be reassembled into the rectangle $ABCD$ shown in Figure 2. The rectangle of Figure 2 has a *base* whose length (AD) equals that of an altitude of our original triangles, and a *height* (AB) whose length equals that of a side of our original triangles. The height of this rectangle is therefore longer than its base.

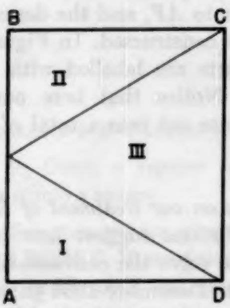


Figure 2

The direction of the solution is now evident: we simply transform rectangle $ABCD$ into a square. To this end, we need a segment whose length is the mean proportional between (AB) and (AD), i.e., the mean proportional between the lengths of a side and an altitude of our original triangles. An easy application of the mean proportional construction serves here. Let us therefore assume that we have at hand a segment whose length is that of a side of the square sought.

On sides AB and CD of our rectangle $ABCD$, mark respectively the point E and the point F such that (AE) and (CF) both equal the length of a side of the desired square. Draw the segment DE, and draw the segment FK perpendicular to CD. Cutting along segments DE and FK, we obtain the six parts shown in Figure 3. Note that each of our two original triangles has now been cut into three parts: Parts 1, 2, 3 comprise one equilateral triangle, while Parts 4, 5, 6 comprise the other equilateral triangle.

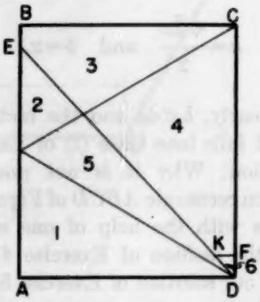
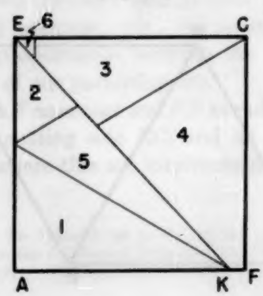


Figure 3

To see how the six parts of rectangle $ABCD$ in Figure 3 reassemble into a square, proceed as follows: Remove the right triangle DFK (the part labelled "6"), and let the portion $BCFKE$ slide down along DE until K coincides with D; lastly, insert DFK so that F coincides with B and FK is an extension of CB. The resulting figure $AECF$, shown in Figure 4, is the square desired.

Figure 4



Remarks about our first solution of Exercise 8: The main step in this solution is the transformation of the rectangle of Figure 2 into the square of Figure 4. The method here is that of the two-cut (three-part) solution of Exercise 3 in our previous paper, viz., that of Case (i) of Exercise 6 (the case in which $b<4a$) in that earlier paper. That this method is the appropriate one is readily seen from the following considerations. Suppose our original equilateral triangles had a side of length x; then their altitudes have length $\sqrt{3}x/2$. The rectangle of Figure 2 therefore has dimensions a by b, where

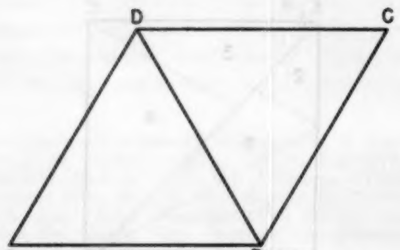
$$a=\frac{\sqrt{3}x}{2} \quad \text{and} \quad b=x.$$

Here, clearly, $b<4a$ and the rectangle of Figure 2 falls into Case (i) of Exercise 6.

Question: Why it is not possible to transform rectangle $ABCD$ of Figure 2 into a square with the help of one stair-case cut, in the fashion of Exercise 4? (Hint: Consult our solution of Exercise 5.)

Solution 2 *of Exercise* 8: Let us return to the two equilateral triangles of Figure 1, but without cutting compose them into the parallelogram $ABCD$ shown in Figure 5. Now transform this parallelogram into a square, using the method of Problem VI in our October 1956 paper. The results appear in Figure 6. The first step

Figure 5



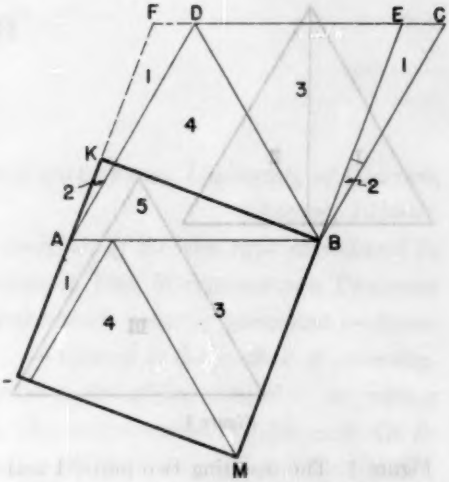


Figure 6

sees parallelogram $ABCD$ transformed into parallelogram $ABEF$, where (AF) is the mean proportional between (AB) and the length of the altitude of $ABCD$ (i.e., the length of an altitude of our original triangles). Next, segment BK is drawn perpendicular to AF, and the desired square $BKLM$ is constructed. In Figure 6, congruent parts are labelled with the same numeral. Notice that here our original triangles are cut into a total of only five parts.

Comments on our treatment of Exercise 8: These solutions suggest how one might proceed to solve the converse of Problem VII in our December 1956 paper, viz., to transform n congruent equilateral triangles into a square; we leave this matter to you. Exercise 8 also suggests a more general problem: *Given two noncongruent equilateral triangles, to transform them into a single square.* A solution of this problem appears in the *American Mathematical Monthly*, LXIII (1956), 667–668. You will find this solution rather unmotivated, and may wish to work out your own, using the natural methods developed in our present series of papers.

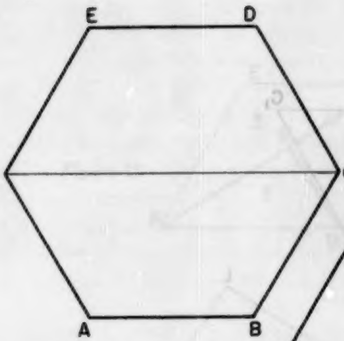


Figure 7

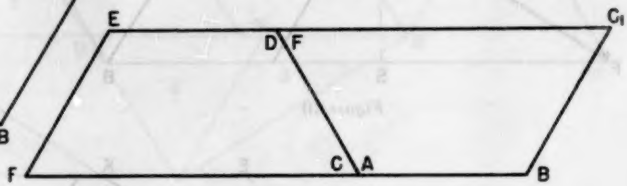


Figure 8

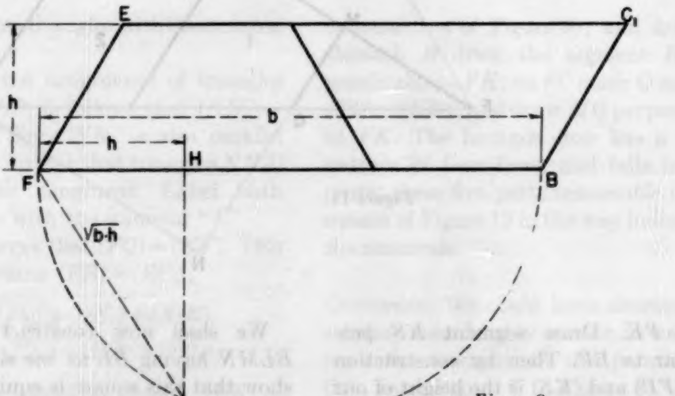


Figure 9

EXERCISE 9 AND ITS SOLUTION

Exercise 9: ***Given a regular hexagon, to transform it into a square.***

Solution: Suppose we are given a regular hexagon $ABCDEF$. If we draw the diagonal FC as in Figure 7, cut along FC and so reassemble the two resulting trapezoids that sides DC and FA are coextensive, then we obtain the parallelogram FEC_1B shown in Figure 8.

Now let us draw a segment whose length equals that of a side of the square sought for. This length, of course, is the mean proportional between the base and the height of the parallelogram shown in Figure 8. There is a familiar way to obtain the desired segment; we propose to construct it in a slightly different way.

Suppose we add to Figure 8, and obtain Figure 9 as follows: Draw the semicircle having FB as diameter. On FB mark the point H such that (FH) is the height of parallelogram FBC_1E. Construct at H the perpendicular to FB and let P be the intersection of this perpendicular with the semicircle, and finally draw segment FP. A known theorem[2] tells us that FP is the desired segment, viz., its length is the mean proportional between the base and height of our parallelogram.

With F as center and FP as radius, draw an arc cutting side EC_1 and let K be the point where this arc intersects EC_1. Draw

[2] See the remarks on p. 445 of our earlier paper "More on the Cutting of Squares," THE MATHEMATICS TEACHER, XLIX (October 1956).

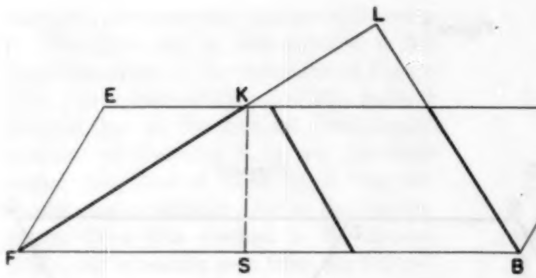


Figure 10

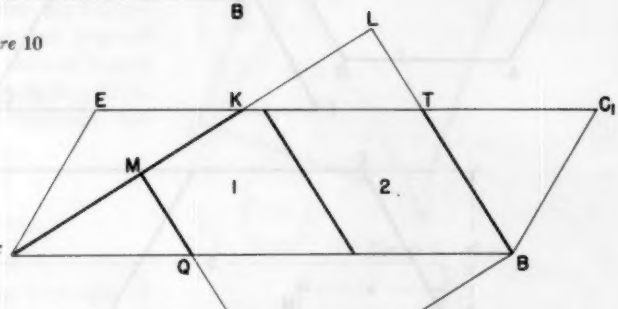


Figure 11

segment FK. Draw segment KS perpendicular to BF. Then by construction $(FK)=(FP)$ and (KS) is the height of our parallelogram, and from Figure 9

$$(1) \qquad (FK)=\sqrt{(BF)\cdot(KS)}.$$

From B draw the perpendicular to FK extended, and let L be the point of intersection. These constructions appear in Figure 10.

Note here that $(BL)=(FK)$. The proof of this fact runs as follows: Right triangles FSK and FLB in Figure 10 have equal corresponding angles, hence are similar; therefore,

$$\frac{(FK)}{(KS)}=\frac{(BF)}{(BL)}$$

and so, using (1),

$$(BL)=\frac{(BF)\cdot(KS)}{(FK)}=\frac{(BF)\cdot(KS)}{\sqrt{(BF)\cdot(KS)}}$$
$$=\sqrt{(BF)\cdot(KS)}=(FK).$$

We shall now construct the square $BLMN$ having BL as one side, and shall show that this square is equivalent to the parallelogram of Figure 8 and hence to our original hexagon of Figure 7. The square is constructed on Figure 10; the result is shown in Figure 11. (We have designated as "T" the point of intersection of BL and C_1E; and as "Q" the point of intersection of MN and FB.) The parts labelled "1" and "2" in Figure 11 are common to square $BLMN$ and parallelogram FBC_1E. Our problem is to show that the remainder of the square (viz., the triangles TLK and QNB) and the remainder of the parallelogram (viz., triangles KEF, FMQ and BC_1T) are composed of congruent parts.

Add to Figure 11 and obtain Figure 12 as follows: From N draw a segment parallel to FE, and let N_1 be its endpoint on BF. (Note that NN_1 is also parallel to BC_1.)

Since $(BN)=(FK)$, and since $\angle BNN_1=\angle KFE$ and $\angle N_1BN=\angle EKF$, we see that triangles EFK and N_1NB are congruent.

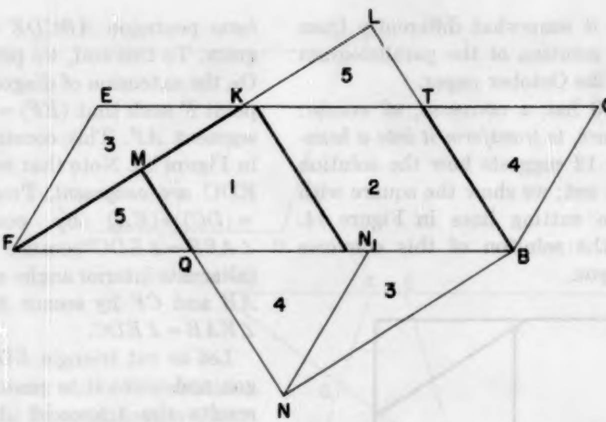


Figure 12

Label both these triangles with the numeral "3."

In view of the congruence of triangles EFK and NN_1B, it follows that $(NN_1) = (FE) = (BC_1)$. Since NN_1 is also parallel to BC_1, we see further that triangles NN_1Q and BC_1T are congruent. Label both these triangles with the numeral "4."

Finally, observe that $(FQ) = (KT)$. This is the case because $(FB) = (EC_1)$,

$$(FB) = (FQ) + (QN_1) + (N_1B),$$

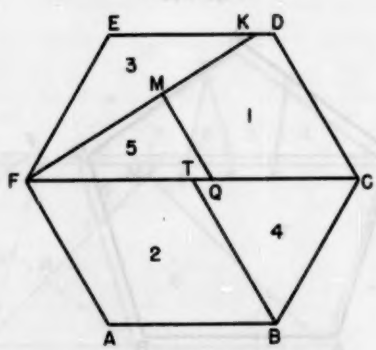
$$(EC_1) = (EK) + (KT) + (TC_1),$$

and from our previous results $(EK) = (N_1B)$ and $(QN_1) = (TC_1)$. Evidently, therefore, the right triangles FMQ and KLT are congruent. Label these triangles "5."

Parts of Figure 12 with the same numeral having been shown congruent, the square and parallelogram are thus equivalent and hence likewise the square and our original hexagon. Of course we do not yet have at hand an explicit solution of Exercise 9; for this, we must construct the cutting lines in our original hexagon. Which is to say, we must construct in Figure 7 the cuts FK, MQ, and BT disclosed in Figure 8. The matter is rather straightforward. We add to Figure 7 and obtain the desired cutting pattern of Figure 13 as follows: Mark K on ED such that (FK) is the length of a side of the desired square (in this connection, e.g., we might use the construction of Figure 9), and draw FK; through B draw the segment BT perpendicular to FK; on FC mark Q such that $(TQ) = (KD)$, and draw MQ perpendicular to FK. The hexagon now has a cutting pattern of four lines, and falls into five parts; these five parts reassemble into the square of Figure 12 in the way indicated by the numerals.

Comments: We could have shortened the solution considerably as soon as we passed from the hexagon to the parallelogram (i.e., from Figure 7 to Figure 8). At that point we could have said simply: transform the parallelogram into a square by the methods of Problem VI in our October 1956 paper. We chose to carry out the solution step by step because we wanted

Figure 13



to organize it somewhat differently from the general solution of the parallelogram problem in the October paper.

Exercise 9 has a converse, of course: *Given a square, to transform it into a hexagon.* Figure 12 suggests how the solution should turn out; we show the square with its requisite cutting lines in Figure 14. We leave the solution of this converse exercise to you.

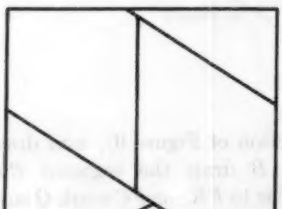


Figure 14

Exercise 10 and its solution

Exercise 10: Given a regular pentagon, to transform it into a square.

Solution: Given a regular pentagon $ABCDE$, we notice in Figure 15 that the diagonal EC divides the pentagon into the isosceles triangle EDC and the trapezoid $ABCE$. It is easy to show (and you should do so) that $ABCE$ is an *equilateral* trapezoid.

The first step in our solution is to transform pentagon $ABCDE$ into a parallelogram. To this end, we proceed as follows: On the extension of diagonal CE mark the point F such that $(EF)=(ED)$, and draw segment AF. This construction is shown in Figure 16. Note that *triangles* AEF *and* EDC *are congruent.* Proof: $(EF)=(ED)=(DC)=(EA)$ by construction, and $\angle AEF=\angle EDC$ because $\angle AEF=\angle EAB$ (alternate interior angles cut from parallels AB and CF by secant AE) and because $\angle EAB=\angle EDC$.

Let us cut triangle EDC off the pentagon and move it to position FEA. There results the trapezoid $ABCF$ shown in Figure 17; this trapezoid is evidently equivalent to (composed of the same parts as) our original pentagon. Now add to Figure 17 and obtain Figure 18 as follows: through midpoint G of side AF draw a segment parallel to BC, and let this segment meet CF at K and meet the extension of AB at L. Now it is evident that triangles ALG and FKG are congruent ($\angle LAG=\angle GFK$, $(FG)=(GA)$, and $\angle KGF=\angle LGA$). If, therefore, we cut off triangle KFG and transpose it to position ALG, we obtain the parallelogram $LBCK$ that can be seen in Figure 18. This parallelogram is evidently equivalent to trapezoid $ABCF$ of Figure 17, and hence to our original pentagon.

From this point, the way to a solution of Exercise 10 is clear: we transform parallelogram $LBCK$ into a square, and then transfer the cutting lines back into our original

Figure 15

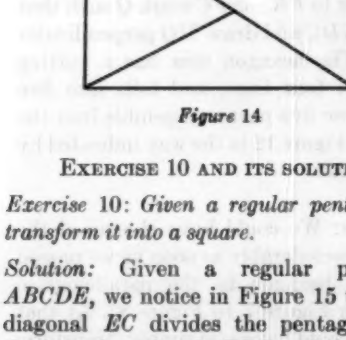
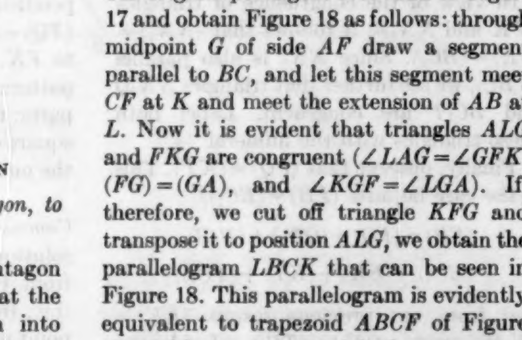


Figure 16



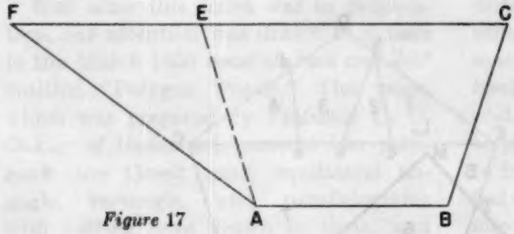
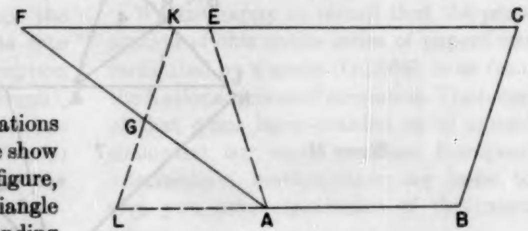


Figure 17

Figure 18



pentagon. Let us carry out both operations simultaneously. To this end, first, we show all the constructions to date in one figure, Figure 19; and then we locate in triangle *EDC* of Figure 19 the cut *ab* corresponding to the cut *GK* in triangle *FEA*. Next, using the method of Exercise 9, let us transform parallelogram *LBCK* into a square. We begin by constructing a segment whose length is the mean proportional between the base and the height of parallelogram *LBCK*. With the help of this segment, we mark on *KC* the point L_1 such that (LL_1) is the mean proportional just determined; we then draw LL_1, take *R* to be the point of intersection of *AF* and LL_1, and take *T*

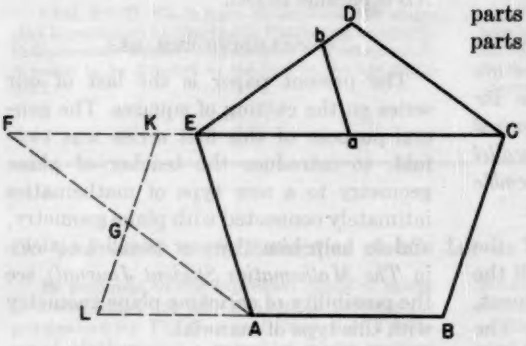


Figure 19

to be the point of intersection of *AE* and LL_1. These constructions are shown in Figure 20. Also shown in Figure 20 are the cut *bc* (in triangle *Eab*) corresponding to cut *LR*, and the cut *de* (in quadrilateral *CDba*) corresponding to cut *RT*. We now draw *BM* perpendicular to LL_1, establish that $(BM) = (LL_1)$—the argument is left to you—and make two final cuts: TL_1 and *BM*. The six cuts *EC*, *bc*, *ab*, *ed*, TL_1, and *BM* divide our pentagon into the seven parts shown in Figure 20. These seven parts can be reassembled into a square,

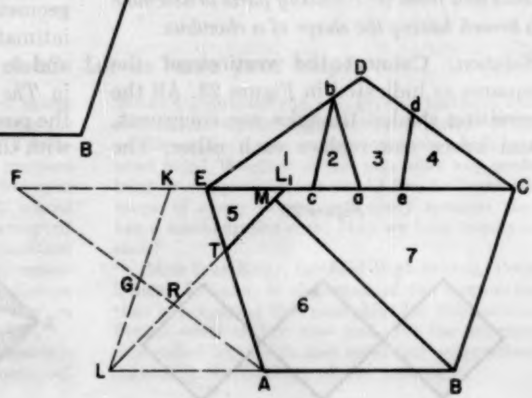


Figure 20

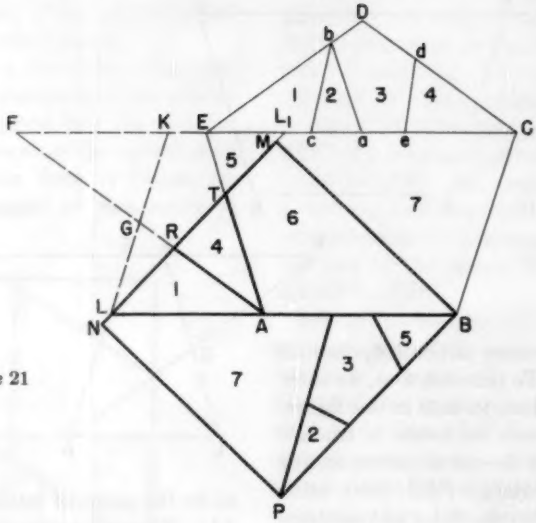


Figure 21

BMNP, as Figure 21 indicates. Our solution of Exercise 10 is complete.

Exercise 11 and its solution

Exercise 10 is the last of our obviously mathematical exercises. We propose to end our paper with a small fillip.

Exercise 11: *A pin consists of three congruent silver squares soldered at the vertices in such a way that the sides of one square are extensions of the sides of another, in the fashion shown in Figure* 22; *it is required to cut this pin along two pairs of parallel lines and from the resulting parts to assemble a brooch having the shape of a rhombus.*

Solution: Connect the vertices of the squares as indicated in Figure 23. All the resulting shaded triangles are congruent, and hence can replace each other. The quadrilateral $ABCD$ which can now be assembled does not have diagonals AC and BD of equal length, but these diagonals are perpendicular and bisect each other. The quadrilateral $ABCD$ is therefore a rhombus, and the four cutting lines are parallel in pairs: AD is parallel to BC, and AB is parallel to DC.

Concluding remarks

The present paper is the last of our series on the cutting of squares. The general purpose of this first series was twofold: to introduce the teacher of plane geometry to a new type of mathematics intimately connected with plane geometry, and to help him (by our parallel articles in *The Mathematics Student Journal*) see the possibility of *enriching* plane geometry with this type of material.

Figure 22

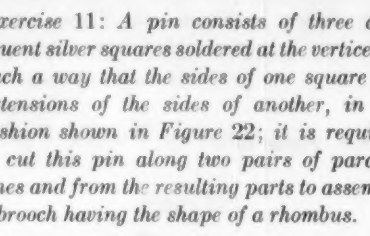
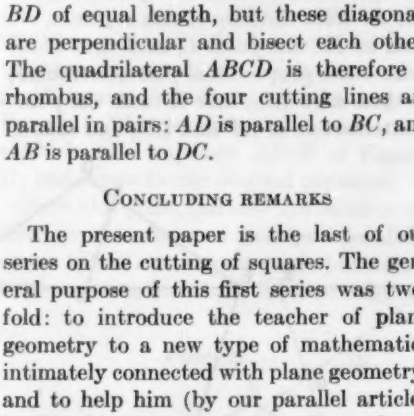


Figure 23



Well after this series was in preparation, our attention was drawn to a page in the March 1956 issue of *Jack and Jill*[3] entitled "Polygon Puzzle." This page, which was prepared by Professor C. O. Oakley of Haverford, presents four polygons (the Greek cross, equilateral triangle, rectangle, and parallelogram) with cutting lines drawn in them, and challenges the reader to cut up the polygons and reassemble the parts into squares. We remark that, with exception of the Greek cross (a nonconvex polygon), the *theory* behind each of these puzzles has been fully exposed in our papers. We also remark that the appearance of the "Polygon Puzzle" just before our co-ordinated series of papers in *The Mathematics Student* and THE MATHEMATICS TEACHER provided an opportunity to "resonate" this material even more widely than planned; we regret missing this opportunity.

The material of this series, along with additional material of a similar sort, is now being revised and reorganized for publication in book form. The Mathematics Staff of the College of the University of Chicago is also giving thought to the problem of distributing this projected book, and others like it,[4] to teachers of high school mathematics; our hope is to make these books available widely, and at minimum cost. In our view it is not sufficient simply to prepare materials that could be useful to high school mathematics teachers; we feel obliged to do all we can to get material *into their hands*.

We are happy to record that the preparation of this entire series of papers was facilitated by a grant (G-2605) to us from the National Science Foundation. The terms of that grant have enabled us to extend somewhat our work on East European intermediate mathematics; we hope to give you other specimens of this work later.

Finally, we express to the editors of THE MATHEMATICS TEACHER and *The Mathematics Student Journal* our appreciation of their kind co-operation in opening their publications to this material and in effecting a useful interplay between the two periodicals. The whole enterprise has been so congenial that we hope to continue it, and to introduce high school mathematics teachers to other material in the same way.

[3] Vol. XVIII, No. 5, pages 34–35. This little magazine is published by the Curtis Publishing Company, Independence Square, Philadelphia 5, Pennsylvania; it seems to be directed to children in the age group 8–12.

[4] Some suggestion of the range of titles in mind here may be gained from the last paragraph (on p. 527) of the paper "Some Remarks on Enrichment," by Izaak Wirszup, THE MATHEMATICS TEACHER, XLIX (November 1956), 519–527.

Do you want a mathematics club? Let us have your help!

In response to the expressed needs of many teachers, steps are being taken to prepare for publication by The National Council of Teachers of Mathematics a pamphlet whose purpose shall be to give assistance to those who have, or would like to consider having, a high school mathematics club. Those who have accepted responsibility for the active work of preparation of this pamphlet believe that one of the essential steps is to secure as complete information as possible concerning mathematics clubs now in existence. Your co-operation is requested.

What suggestions have you for materials that should be included in such a pamphlet? What suggestions can you give to teachers who are considering the organization of a club? What problems have you encountered on which you need help? Whether or not you have any problems or suggestions, we would like to know the name of every school and every sponsor that has a mathematics club. May we hear from you soon?

Miss Inez Kelly, Garfield High School, Terre Haute, Indiana, is chairman of the committee that is preparing this pamphlet for publication. Please write to her now and give the information called for above and make any suggestions regarding the content of the pamphlet.

Right answer—wrong solution

BENJAMIN BRAVERMAN, *Abraham Lincoln High School, Brooklyn, New York.*

Can a pupil come up with the right answer to a problem despite numerous fundamental errors in his solution?

All teachers of mathematics have experienced any number of cases where a pupil makes one error, and then a second one which compensates or cancels out the first error. In such cases, teachers usually find it hard to convince the immature pupil that he is not entitled to full credit for the resulting solution.

It remained, however, for a pupil in our school to come up with something that not even the oldest and most experienced teachers in the department had ever come across. In a solution to a trigonometric problem he was guilty of at least eight independent errors, many of them fundamental. Nevertheless, by complying with the degree of accuracy required in the problem he was solving, he arrived at the correct answer.

This happened in June 1956, when the pupil was taking the state, or Regents, examination in Eleventh Year Mathematics. He was answering Question 32 in that examination, which reads as follows: In triangle ABC, $b=16$ feet, $c=14$ feet, and angle $A=136°28'$. Find a to the nearest foot.

Quite properly, the pupil attempted to solve the problem by the Law of Cosines. A copy of his solution is shown in Figure 1. The reader is advised to study it carefully before continuing with this article. He will otherwise miss the mental stimulation of discovering the numerous errors for himself.

Now for a summary of all the errors—and the writer hopes that none escaped the careful reader.

1. The pupil starts with an incorrect version of the Law of Cosines. He uses $+2bc \cos A$, when he should have used $-2bc \cos A$.
2. He combines the unlike terms $196+256+448 \cos A$ into $900 \cos A$.
3. In reducing the cosine of $136°28'$, he uses $90°$ as his base. However, he takes the cosine of the resulting acute angle, $46°28'$, instead of its sine.
4. He makes the cosine of an angle in the second quadrant positive when it should be negative.
5. In looking up the cosine of $46°28'$, he uses the logarithmic table of the trigonometric functions instead of the table of the natural values of these functions.
6. In using the logarithmic table, he completely ignores the characteristic and uses the mantissa only.
7. In his interpolation for the $8'$, he subtracts the calculated difference of .3 from the mantissa 8391 instead of adding it to the mantissa 8378.
8. He makes 28 the nearest integral square root of 754.9, because in squaring 27, he incorrectly made his first partial product 169 instead of 189, thus making the square of 27, 709 instead of 729.

To save the reader time and trouble, the correct solution to the problem follows: $a^2=776.8$. Hence, $a=27.9$, and a is therefore 28, to the nearest foot.

Isn't it truly amazing that a pupil after making eight errors, most of which are fundamental, could have come up with the correct answer?

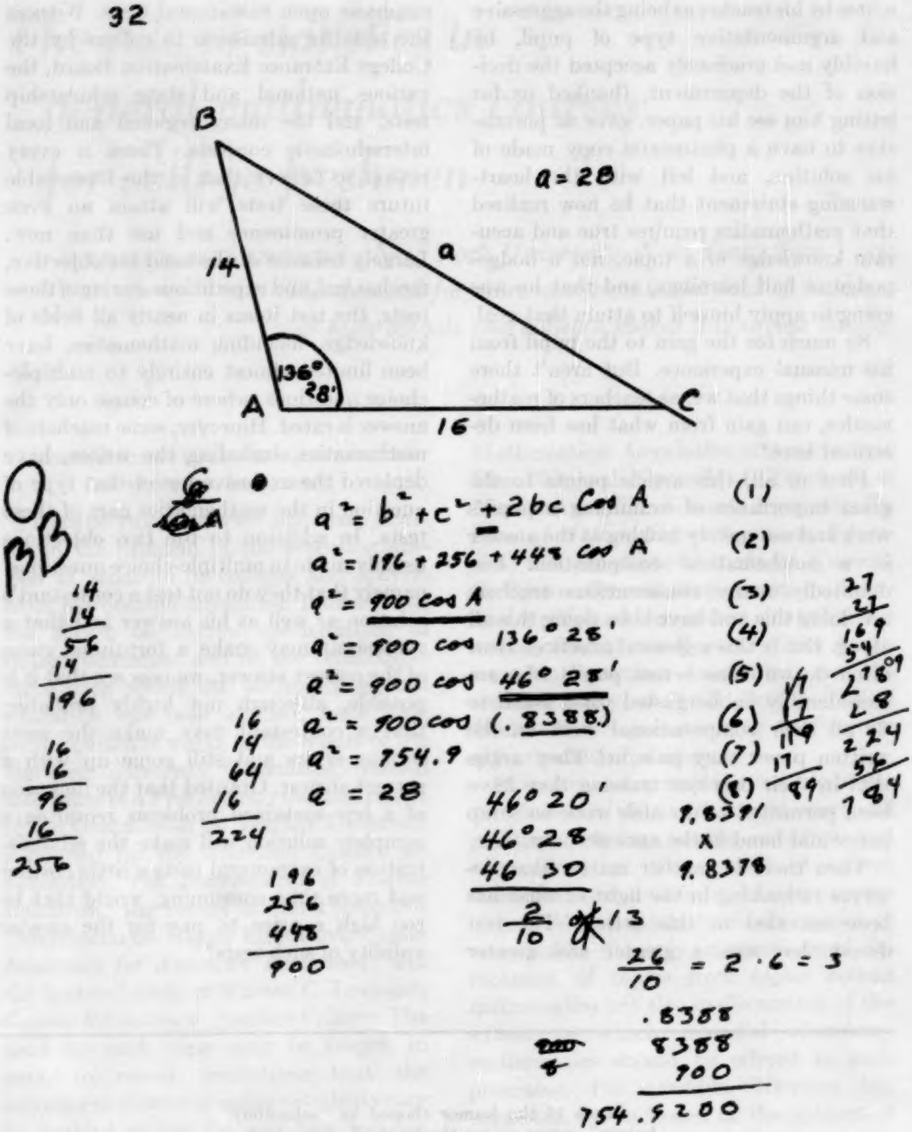


Figure 1

I shall leave it to one who is more gifted than the writer in the mathematical theory of probability to compute the odds that were against this pupil's producing the correct answer, despite his eight errors. Be that as it may, the reader will note that the pupil did not receive any credit for his solution. A few days after the examination, the pupil was invited to the department office and allowed to see his paper. The eight errors were pointed out to him. Although he had been reported to the

writer by his teacher as being the aggressive and argumentative type of pupil, he humbly and graciously accepted the decision of the department, thanked us for letting him see his paper, gave us permission to have a photostatic copy made of his solution, and left with the heartwarming statement that he now realized that mathematics requires true and accurate knowledge of a topic, not a hodgepodge of half learnings, and that he was going to apply himself to attain that goal.

So much for the gain to the pupil from his unusual experience. But aren't there some things that we, as teachers of mathematics, can gain from what has been described here?

First of all, this article points to the great importance of examining a pupil's work and not merely looking at the answer in a mathematical computation. Undoubtedly many conscientious teachers are doing this and have been doing this all along. But is this a general practice? How often do we come across pupils who are considerably flabbergasted when asked to do all their computational work on the written paper they pass in? They argue that in their previous training they have been permitted to do such work on scrap paper and hand in the answers only.

Then there is another matter that deserves rethinking in the light of what has been revealed in this article. The last decade has seen a greater and greater emphasis upon extramural tests. Witness the tests for admission to college by the College Entrance Examination Board, the various national and state scholarship tests, and the many regional and local interscholastic contests. There is every reason to believe that in the foreseeable future these tests will attain an even greater prominence and use than now. Largely because of the need for objective, mechanical, and expeditious scoring of these tests, the test items in nearly all fields of knowledge, including mathematics, have been limited almost entirely to multiple-choice questions, where of course only the answer is rated. However, some teachers of mathematics, including the writer, have deplored the exclusive use of that type of question in the mathematics part of these tests. In addition to the two objections usually made to multiple-choice questions, namely that they do not test a contestant's solution as well as his answer and that a contestant may make a fortuitous guess of the correct answer, we now see that it is possible, although not highly probable, that a contestant may make the most serious errors and still come up with a correct answer. Granted that the inclusion of a few sustained problems requiring a complete solution will make the administration of extramural tests a little costlier and more time consuming, would that be too high a price to pay for the greater validity of such tests?

Much of the humor classed as "schoolboy boners" comes under the head of wrong or inadequate concepts. The boy who thought an average was "something hens lay eggs on" (he had read the arithmetic problem: Farmer Jones' hens laid 24 eggs on Sunday, 32 eggs on Monday, 28 eggs on Tuesday, etc. How many eggs did they lay on the average during the week?) was suffering from an inaccurate concept.—*Taken from David H. Russell,* Children's Thinking.

The role of statistics in general mathematics courses for college freshmen

HERMAN ROSENBERG, *New York University, New York, New York.*

The demand for more attention to the field of statistics in high schools and colleges makes this article timely.

Introduction: College Mathematics in General Education

WITHIN THE PAST TWO DECADES, interest in the problem of the *general* education of students in colleges and universities has become intensified. Indicative of the attack on narrow specialization in college education and of the remarkable vigor with which the subject of general education has been studied, have been such significant works as the report of the Harvard University Committee, *General Education in a Free Society;* the final report of the Commission on Teacher Education appointed by the American Council on Education, *The Improvement of Teacher Education;* the report of the President's Commission on Higher Education, *Higher Education for American Democracy;* and the doctoral study of Warren C. Lovinger, *General Education in Teachers Colleges.* The need for such vigor may be traced, in part, to recent predictions that the current enrollment of college students may be doubled within the next two decades.

Mathematics educators have hardly been idle in evidencing interest in the more restricted problem of the role of college *mathematics* in programs of general education. One has only to recall such early contributions to the problem as the report of the Progressive Education Association, *Mathematics in General Education;* the final report of the Joint Commission of the Mathematical Association of America and The National Council of Teachers of Mathematics, *The Fifteenth Yearbook: The Place of Mathematics in Secondary Education;* and the doctoral study of Kenneth E. Brown, *General Mathematics in American Colleges.* More recent contributions include the "Duren Report" of the American Mathematical Association and the research study of R. P. Bentz, *Critical Mathematics Requirements for the Program of the Community College.*

Even a casual survey of available literature indicates wide variation in views and practices relating to the selection and organization of the content of college mathematics courses in freshman programs of general education. Such divergence of views exists not only in questions of the inclusion of topics from *higher* college mathematics but also in discussions of the extent to which remedial *elementary* mathematics should be offered in such programs. For example, Newsom has suggested the inclusion of the subject of permutations in such courses,[1] while Northrop has objected to such inclusion on the ground of nonutility.[2] On the one

[1] Carroll V. Newsom, "A Course in College Mathematics for a Program of General Education," THE MATHEMATICS TEACHER, XLII (January 1949), 20.

[2] E. P. Northrop, "The Mathematics Program in the University of Chicago," *American Mathematical Monthly*, LV (January 1948), 4.

hand, Earl J. McGrath has suggested that in such courses, elements of trigonometry be illustrated in surveying.[3] Newsom, however, would omit such surveying applications on the ground that they have little value for most people. He (like Northrop[4] and Ore[5]) would replace these applications with applications of trigonometry to such important periodic phenomena as radio, light, and music.[6] Further illustration of such divergence may be found in the conflicting views on remedial mathematics[7] at the college level. The report of the Harvard Committee opposes such "college repair" on the ground that remedial mathematics would ". . . necessarily involve much simple repetition of work already done in secondary school and would offer little hope that such a second exposure would result in substantial educational gain."[8]

The need for general statistical education

The major purpose of this article is to concentrate on the special problem of the role of *statistics* in the program of general mathematics for college freshmen. Should statistics be included in such courses? Current views appear to favor quite strongly the inclusion of elements of statistics in general education. Strong support for such inclusion has come from such groups as the Joint Commission of the Mathematical Association of America and The National Council of Teachers of Mathematics,[9] the Harvard University Committee,[10] and McGrath and others,[11] and from leaders in the field of mathematics education.

How may such support for statistics in general college education be explained? Certainly statistical training may be helpful in meeting the specialized *vocational* needs of college students interested in careers in fields such as engineering, business, education, government, psychology, the social sciences, the natural sciences, industry, medicine, mathematics, etc. Thus, the recent advances in industrial quality control, business statistical accounting, automation, applications of statistics in economics and sociology, and mathematical statistics[12] have strengthened the support for vocational statistical education.

The fact that statistics plays a significant role in such a large number of important occupations for college-trained students has influenced the support for statistics in college education. However, it would seem that a basic justification for inclusion of statistics in a general college program should be found in the possible contributions of statistics to the *general* needs of college students as citizens of a democratic society in which the consumption of statistical data is an important element in the processes of communication. American citizens, for example, may have considerable interest in the "consumption" of recent statistical data concerning possible relations between cigarette smoking and lung cancer.

Such considerations may illuminate (1) the reason for the unhappiness of Burr in finding that a statistical background for high school and junior college students is unfortunately absent or forms only a

[3] Earl J. McGrath and others, *Toward General Education* (New York: Macmillan Company, 1948), p. 105.

[4] Northrop, *op. cit.*, p. 4.

[5] Oystein Ore, "Mathematics for Students of the Humanities," *American Mathematical Monthly*, LI (October 1944), 456.

[6] Newsom, *op. cit.*, p. 23.

[7] McGrath and others, *op. cit.*, p. 25, favor "repair."

[8] Harvard University Committee, *General Education in a Free Society* (Cambridge, Massachusetts: Harvard University Press, 1945), p. 224.

[9] National Council of Teachers of Mathematics, *The Fifteenth Yearbook: The Place of Mathematics in Secondary Education* (New York: Bureau of Publications, Teachers College, Columbia University, 1940), pp. 159–161.

[10] Harvard University Committee, *op. cit.*, pp. 160–161.

[11] McGrath and others, *op. cit.*, p. 105.

[12] For a recent article on the modern role of statistics in contemporary society, see, for example, Richard S. Burington, "Contemporary Applications of Mathematics," The Mathematics Teacher, XLIX (May 1956), 322–329.

minute part of the curriculum,[13] and (2) the plea of Newsom that a "... program in general education can no longer ignore the universal acceptance of statistics as fundamental in almost every phase of human endeavor."[14]

Problems of content in general statistical education

If it is assumed that statistics should be included in general college education, then there still remain such crucial problems as the objectives of instruction and the precise content to be included. Certainly, the needs of *general* students as *consumers* of statistical data indicate that considerable attention should be given to the *analysis* and *interpretation* of statistical literature rather than solely to *techniques* of statistical computation. Furthermore, the statistical part of the general mathematics college course should develop not only facility in performing basic computations and other procedures of statistics, but also an *understanding* of the logic of such procedures. Here, as in the rest of the mathematics program for a democratic society, mathematical content should be taught "... in the light of modern pedagogical theories of meaning and understanding rather than as blinding, soulless, meaningless mechanics."[15]

How much statistics should be included in general mathematics courses for college freshmen? Considerable differences in opinion also exist here. Several of the recent general mathematics textbooks[16] for college freshmen devote about one chapter to an introduction to statistics. In his concern over the neglect of statistics, Allendoerfer has suggested an entire semester of statistics in the college mathematics program for general education.[17] Many mathematics educators, however, would hesitate to devote such a large part of a two-semester mathematics course in general education to one area. In this connection, it may be of interest to note Beatley's recent observation of trends in freshman mathematics toward a "marked increase in instruction in statistics"[18] and toward the replacement of some freshman calculus with statistics for students mainly interested in the social rather than the natural sciences.[19]

What specific basic topics from statistics should be included in the general college mathematics course? Because of existing differences in local and individual needs, it would be inadvisable to advocate a completely uniform content for all courses. However, in view of the interest in such courses in the *general* education of students for citizenship, some attention might well be given to the nature and dangers of statistical misrepresentation. Support for this is not difficult to locate in mathematical literature. Thus, Newsom has warned that "... the conclusions that are being foisted upon an innocent public by some so-called statisticians are so unjustified that every elementary presentation of statistics should spend some time on fallacious statistical reasoning."[20] Burr has stressed that the fundamental objective is to ensure that students will never again look at a number with the same innocent, trusting confidence which they revealed in their previous school years.[21]

[13] Irving W. Burr, "What Principles and Applications of Statistics Should be Taught in High School and Junior College?" The Mathematics Teacher, XLIV (January 1951), 10.

[14] Newsom, *op. cit.*, p. 22.

[15] Herman Rosenberg, "Higher Dividends with Series of Higher Orders," *The New Jersey Mathematics Teacher*, VIII (May 1952), 15.

[16] See, for example, such textbooks as C. B. Allendoerfer and C. O. Oakley, *Principles of Mathematics* (New York: McGraw-Hill Book Company, 1955); J. Houston Banks, *Elements of Mathematics* (New York: Allyn and Bacon, Inc., 1956); John E. Freund, *A Modern Introduction to Mathematics* (Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1956); and Thomas L. Wade and Howard E. Taylor, *Fundamental Mathematics* (New York: McGraw-Hill Book Company, 1956).

[17] C. B. Allendoerfer, "Mathematics for Liberal Arts Students," *American Mathematical Monthly*, LIV (November 1947), 574–577.

[18] Ralph Beatley, "Desirable Alterations in Order and Emphasis of Certain Topics in Algebra," The Mathematics Teacher, XLIX (May 1956), 361.

[19] *Loc. cit.*

[20] Newsom, *op. cit.*, p. 22.

[21] Burr, *op. cit.*, p. 12.

Appropriate illustrations of pitfalls in statistical reasoning may be provided throughout the introduction to statistics. Thus, in discussing the collecting, tabulating, and graphic techniques involved in the organization of statistical data into frequency distributions, attention might be called to the misuse of the geometric concept that the ratio of the corresponding heights of two similar figures may not equal the ratio of the areas of the figures. In the study of the descriptive measures of statistical data, such as the mean, median, and mode measures of central tendency, students may be led to discover the inadequacy of using the arithmetic mean as an appropriate measure of central tendency for a distribution where extreme measures would have undue weight on the mean. Thus, students may think twice when they read that an employer, utilizing the arithmetic mean in an attempt to justify a current wage scale for "average workers," includes among those "average workers" certain managerial personnel whose unusually high wages may be far from average! In introducing the topic of the normal probability curve, students may learn that the use of such a curve for grading students in a "nonnormal" class may lack mathematical justification. Similar illustrations may be utilized in treating such topics as the measures of variability (especially the range and standard deviation), correlation (so frequently confused with causation), and statistical sampling (including tests of hypotheses).

The selection of specific topics and the extent of study of each topic in the general mathematics courses will depend upon many factors. One very important factor is the mathematical maturity of the members of the group being taught. Some restriction on the amount of mathematical development of such a course may exist where the mathematical prerequisites for the course are relatively low. Since general education should not ignore the implications of the special or vocational plans of students, such general mathematics courses may be somewhat slanted to meet the special needs of various groups. For example, where such courses are given in schools of commerce as part of the general education of business students, some time might conceivably be given to the subject of index numbers and time-series analysis. Where such courses are given in teachers colleges as part of the general education of preservice teachers, some attention might properly be given to the subject of percentiles. (The problem, however, is somewhat complicated by the fact that students in many teachers colleges do not intend to teach, but, nevertheless, seek their preprofessional training and liberal arts degrees from such teachers colleges.) In the final analysis, the major factor in determining the role of statistics in general mathematics college courses may be the extent of the statistical interest and background of a rather key person in mathematics education—the mathematics teacher.

A positive approach to the teaching of math

My experience has taught me it is the techniques of mathematics that are sometimes extremely distasteful for certain types of students. Give these students an opportunity to see the thinking, the logic, involved in doing mathematics, and their attitudes toward the subject change. The gain is a larger number of people who are able to work intelligently with mathematics, a field which seems to be creeping into almost every area. Those students who are interested in mathematics as such or in engineering or in any of the sciences will come to their more advanced work with a conviction that mathematics is a way of approaching problems, one that is not a stereotype of the distance-rate-time, mixture, digit, or work sort.—*Averill M. Chapman, in the* California Journal of Secondary Education.

A new approach to high school mathematics

E. J. COGAN, *Dartmouth College, Hanover, New Hampshire.*

These are days in which many are seeking a new curriculum pattern in mathematics. In broad outline, this article proposes a new pattern for your consideration.

A STATEMENT often heard during the past ten years is that the teaching of mathematics at the college level is antiquated by at least fifty years. Reforms at this level have been suggested from time to time, and many of these are being investigated. Perhaps the single greatest handicap to such a program is that it requires a corresponding reform in the teaching of high school mathematics. Without this reform at the high school level, the new college program will attain general applicability only with great difficulty. To be sure, reforms at the secondary level are being investigated on a small scale. It is the purpose of this paper to suggest an approach to these problems and to investigate possible advantages and difficulties of a suggested program.

Deficiencies in the current high school program

Perhaps the most striking criticism that can be made of the present program in high schools is that it fails to capture the imagination of the students. Courses in algebra are likely to consist only of techniques of simplifying expressions and solving equations. Geometry courses usually plod through one proof after another without sufficient motivation. Trigonometry courses emphasize the solving of triangles and graphing instead of analytical trigonometry. Calculus tends to mean the memorization and use of rules that are largely algebraic in flavor. The student is easily bored with learning one technique after another without knowing why these techniques are important or to what kinds of everyday problems the techniques may be applied. Mathematics becomes, at this level, the equivalent of a dead language that no one has any use for and that is learned for the sake of mental discipline, as our grandfathers conceived it.

Prospective teachers of high school mathematics are not made sufficiently aware of basic concepts in their field because not enough time is allowed them in their course work to pursue such studies. The consequence of this is that they are in general inefficient at motivating the student in his studies and ineffective at satisfying those students whose questions show more than the usual depth and interest. Students with exceptional abilities in science, engineering, and mathematics will tend to shy away from these fields unless their quest for understanding is satisfied. An attempt to endow the average student with a deeper understanding of mathematics is very likely to increase both his appreciation and his aptitudes in the subject. This can be done only when the teacher is able to explain the relations between mathematics and the pursuit of life. These relations can be drawn most readily within the foundations of mathematics.

The power of mathematics in fields different from science is not sufficiently emphasized. Such ideas as models for social systems, mathematical approaches to

games, the power of computers and other automata would captivate the student's interest and assign more importance to mathematics as a general, rather than a specific, pursuit. The structure of number systems and their origin would tend to invalidate the belief that mathematics is "two plus two equals four, and that's that."

A NEW APPROACH FOR THE STUDENT

Mathematics curricula for high schools oriented toward an understanding of basic ideas have been formulated in detail by various groups considering the problem. The purpose of this section is to furnish general suggestions as to the content of a new curriculum and to justify this content in various ways from the point of view of the student. The implementation, justification, and problems involved from the teacher's viewpoint will be discussed in the next section.

A number of assumptions about the capabilities of both students and teachers underlie the suggestions below. First, the most efficient way to motivate students in mathematics is to educate their intuitions in the field so that the intimate connections between mathematics and everyday living become apparent, thus rendering the material less sterile. Even the most tenuous connections with what is familiar and important serve to heighten the student's interest. Second, the most efficient way to accomplish this is to present the most basic mathematical notions at an early stage. Third, at the level suggested, the material is not beyond the abilities of the average student. Indeed, introducing such material is likely to enable more students of average ability to develop above average taste and ability in mathematics. Fourth, with appropriate preparation and guidance, the teaching of such material is well within the abilities of most teachers trained in mathematics. Fifth, using basic concepts, the introduction of many of the essential techniques may be taught more speedily, thoroughly, and efficiently, and it is likely that the student will be able to retain them for a longer time. Sixth, the new approach will enable the prospective college student to start mathematics at a higher level as well as provide him with cultural stimuli that would otherwise have been unavailable to him. Seventh, the student who does not continue his education past high school would share in the cultural advantages of the new approach without being deprived of techniques which may prove useful to him later on.

The list below contains the basic elements of this approach.

1. The historical development of the number system. Different bases for notation. The need for different kinds of numbers.
2. The operations of arithmetic. The relations of arithmetic.
3. Approximations and "number sense."
4. Euclid's axioms for geometry. The parallel axiom, the non-Euclidean geometries, and their effect on mathematical thought.
5. The role of axioms in mathematics. Self-evidence versus instrumentality. Formalism and abstraction.
6. Calculus of propositions and truth tables. Predicate calculus and the role of quantifiers in everyday use as well as in mathematics.
7. Rules and methods of proof in mathematics. Dependence of mathematical techniques upon axioms and methods of proof.
8. Intuitive set theory, its relations and operations. Measures on sets.
9. Relations and Functions.
10. The number system re-examinde from a slightly more formal point of view.
11. Algebra: equations and inequalities.
12. Direct and indirect measurement. Analytic trigonometry.
13. The real line and the co-ordinate plane. Analytic geometry of the line.
14. Examples of analytic and synthetic proofs of well-known geometric properties.
15. The notion of a limit and why it is

considered. More on approximations.

16. Average rates of change and derivatives.

17. Limits of sums and definite integrals.

18. Indefinite integrals and differential equations.

19. Probability and counting.

20. Empirical induction and statistics.

It would be well to remark that, even though this list seems quite high flown at first glance, the technical material need not be any more complicated than that which is presented in current curricula. What is important is that the basis be so well laid that the techniques presented follow simply and obviously from ideas that are well anchored to the real and familiar. Note also that the list is flexible both in its order and in that many of the topics contained in it may be omitted for certain classes of students, in particular for those not entering college, and for those not planning to take mathematics in college. Inspection of some of the books listed in the bibliography will convince the reader that problem work need not be difficult for the thoughtful student.

A NEW APPROACH—THE TEACHER'S POINT OF VIEW

It may well be objected that there are not enough teachers equipped for the task outlined. Yet there are means by which an increase in the number of such teachers may be brought about. First, of course, among these is the provision of time and opportunity for the introduction of courses in basic mathematics in teacher training institutions. Steps in this direction are being already taken and each year more colleges are adding such courses to their curricula. In many cases, a teacher can get enough background in a semester, and in nearly all cases in a year. College level mathematics courses are being reconsidered and, in more and more cases, oriented to the modern approach. Second, much background has been and will continue to be communicated to teachers at summer institutes held throughout the country. These institutes provide discussions of new methods and new approaches to teaching and opportunities for high school teachers to exchange among themselves and with college teachers fertile and important new ideas.

Third, the problem of producing textbook material along the lines suggested must be investigated. For this type of material, texts just as useful to the teacher as to the student can be written. The texts listed below will give the reader an idea of the work that has been done at the college level. A look at some of these texts will certainly provide a freshness and novelty to future writers of high school texts.

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During the 40-year period from 1910 to 1950 the population of the United States increased 65 per cent while the number of college students increased almost 650 per cent.

A club project in a modern use of mathematics

WALLACE MANHEIMER, *Franklin K. Lane High School, Brooklyn, New York.*

This is a description of some interesting activities for club meetings.

SINCE LAST SEPTEMBER the mathematics club at our school has undertaken what proved to be an instructive and entertaining study of the mathematics underlying the digital computer. In the hope that it may be of help to other schools trying a similar project, I should like to outline the program we followed.

A large number of students were attracted to the opening meeting of the club by our announcing that it would feature mathematical card tricks. The reader can find some of these tricks in an interesting free pamphlet called "Take a Card," issued by the Association of American Playing Card Manufacturers.[1]

Two of the tricks served later to show the analysis of a routine into a logical flow chart as a preliminary to coding for a digital computer. They were each of the "mind reading" variety. In the first one, sixteen cards were arranged in a rectangle, a card was picked by the audience, and the "mind reader" was summoned by a sentence. He then selected the proper card. The reader undoubtedly has seen this trick or similar ones many times. The sentence calling the mind reader was in the form "John likes spinach," the subject and predicate each being human, animal, vegetable, or mineral. The cards were mentally subdivided in groups as shown in Figure 1.

The opening of the sentence gave the mind reader information about the group containing the card and the end of the sentence told him the particular card in the group. For example, "John likes spinach" gives the information, "First group, third card."

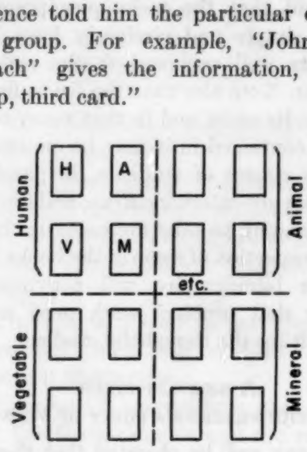


Figure 1

The second trick, which was much more important in the development of the main objective, proved to be very entertaining to the club. It was announced that one of our teachers, who was in an office at the other end of the building, was a "wizard" and could name any card chosen by a student. When the card was selected, the wizard was called on the school telephone by the faculty adviser, the conversation at the club end going as follows:

"Hello, is the wizard there?"

Then:

"Hello, wizard?"

and finally:

"Hold the wire, please."

[1] *Take a Card*, Association of American Playing Card Manufacturers, 420 Lexington Ave., New York 17, N.Y.

The student was handed the telephone and was confounded to hear his card read off at the other end of the wire. The trick was performed several times, and many students made ingenious guesses as to how it was done. Solutions to all the tricks were promised for a subsequent meeting of the club.

At the second meeting several pictures of digital computers were shown, and articles were read describing their remarkable powers. Some of the social effects of automation and allied advances were discussed. It was then pointed out that the basis of the computer is simply whether the current is on or off in an electrical circuit. The relationship of this to a "yes or no" answer to a question was shown, and the concept was developed that a chain of routine thinking, however complicated, can be broken down into a sequence of yes-no responses. As a partial illustration a game of Twenty Questions was played by two students. Club members were challenged to show how they could find a previously selected number from 1 to 1000 in exactly ten questions with yes-no answers. Several students hit upon the plan of successively bisecting the interval, and their method was saved for illustrative material in the next meeting. Finally the question was raised as to how a machine engineered to give only yes-no answers could conveniently solve problems in arithmetic, in which we have ten digits.

The third club meeting, entitled "Yes-No Arithmetic," was devoted to an exposition of binary numbers. A table of binary numbers from 1 to 111111 (63) was set up and a few addition and subtraction examples were solved. The powers of binary numbers were chosen by a series of games and tricks. The first was the familiar one of "age cards," which are illustrated below.

A student was asked to pick the cards upon which his age appeared, and the age was found by adding the first numbers of each of the selected cards. The club members were unable to find the principle upon which the cards were constructed until all the numbers were expressed in the binary system. Then it became plain that the "yes or no" choice of each card selected a "1 or 0" digit in the binary expression of the age. A similar procedure was shown in the process of guessing any number from 1 to 1000 in ten yes-no type questions.

1	33
3	35
5	37
7	39
9	41
11	43
13	45
15	47
17	49
19	51
21	53
23	55
25	57
27	59
29	61
31	63

2	34
3	35
6	38
7	39
10	42
11	43
14	46
15	47
18	50
19	51
22	54
23	55
26	58
27	59
30	62
31	63

4	36
5	37
6	38
7	39
12	44
13	45
14	46
15	47
20	52
21	53
22	54
23	55
28	60
29	61
30	62
31	63

8	40
9	41
10	42
11	43
12	44
13	45
14	46
15	47
24	56
25	57
26	58
27	59
28	60
29	61
30	62
31	63

16	48
17	49
18	50
19	51
20	52
21	53
22	54
23	55
24	56
25	57
26	58
27	59
28	60
29	61
30	62
31	63

32	48
33	49
34	50
35	51
36	52
37	53
38	54
39	55
40	56
41	57
42	58
43	59
44	60
45	61
46	62
47	63

Next, a set of punched age cards was shown in which yes or no answers were indicated by not inverting or inverting a card. In this trick the proper age showed through a series of punches when the cards were smoothed out.

The last device was an entertaining variant of the game of "nim." An arrangement of dashes was set up on the blackboard as follows:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & 1 & 1 & & \\ & 1 & 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}$$

Two players alternate in crossing out any number of dashes in a single row. The player who crosses out the last dash is the winner. After a discouraging series of games with the faculty adviser, the club was shown some rough and ready rules of play, and then the underlying principle was revealed by expressing the numbers 1, 3, 5, and 7 in the binary system:

$$\begin{array}{rrr} & & 1 \\ & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{array}$$

The second player must always win if he keeps the 1 digits matched in each column.

At the fourth club meeting, the solutions to the card tricks were given. A "thought diagram" for the human-animal-vegetable-mineral card trick was drawn as seen in Figure 2. Students had no difficulty in understanding the connection between the diagram and the routine of performing the trick.

Next the "wizard" trick was explained by revealing the other end of the telephone conversation. At the first question the wizard slowly recited, "Ace, two, three, four, ..." When he reached the number on the card chosen by the student, the faculty adviser interrupted him with the statement, "Hello, wizard?"

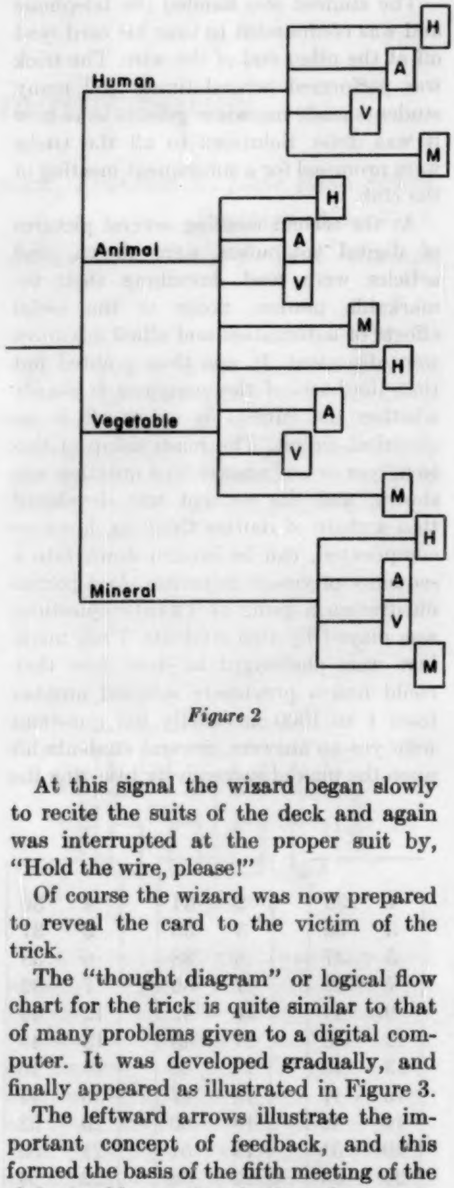


Figure 2

At this signal the wizard began slowly to recite the suits of the deck and again was interrupted at the proper suit by, "Hold the wire, please!"

Of course the wizard was now prepared to reveal the card to the victim of the trick.

The "thought diagram" or logical flow chart for the trick is quite similar to that of many problems given to a digital computer. It was developed gradually, and finally appeared as illustrated in Figure 3.

The leftward arrows illustrate the important concept of feedback, and this formed the basis of the fifth meeting of the club. At this meeting the flow charts were reviewed and they were shown to illustrate the difference between a nonfeedback and a feedback routine. The occurrence of feedback in biology, psychology, electronics, and politics was shown. Some of Norbert Wiener's penetrating comments on feed-

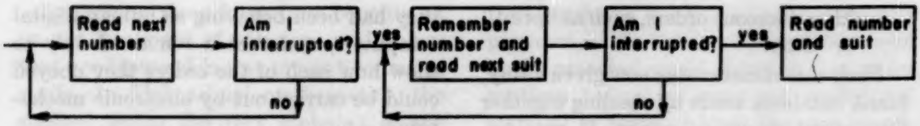


Figure 3

back were read. Several experiments were performed. For example, a student was asked to close his eyes and stand motionless on his toes. He was then asked to describe the feedback nature of his maintenance of balance.

The sixth club meeting saw the analysis of a card sorting problem. A logical flow chart was set up, and some of the engineering features of modern card sorters were explained. Fortunately, the programming at our school is done by means of a device using punched cards and long needles, and many of our club members are on the program committee. We are also fortunate in New York in being close to the headquarters of I.B.M., which offers a splendid tour of their equipment for interested organizations. The club is planning to take advantage of this some time next term. Students were told that at the next meeting they would finally make the acquaintance of a digital computer, although perhaps not exactly in the way they might expect.

At the seventh meeting a problem in long division was solved by a method of successive subtractions, as illustrated below:

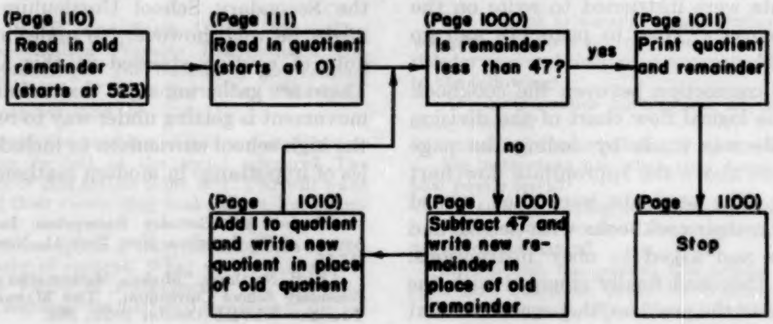
523 (dividend)
<u>47</u> (divisor)
476 (1st remainder)
<u>47</u>
429 (2nd remainder)
. . .
. . . etc.

It was seen that the numbered place of the first remainder smaller than 47 is the quotient, and its value the true remainder. A logical flow chart (Figure 4) of the routine was developed.

The club members were now asked to imagine a loyal, patient, and slow-witted human servant who could keep only one order at a time in his mind, and who was capable of following orders only of the type indicated in the flow chart. These orders were summarized as follows:

1. Writing a number or erasing it (setting orders).
2. Adding or subtracting (arithmetic orders).
3. Seeing if one number is larger than another (comparing orders).

Figure 4



4. Miscellaneous orders, such as "print" or "stop."

Each club member was now given a tiny, blank notebook made by stapling together four small sheets of paper. It was announced that this was to become a 13-page "cookbook" for solving the problem, with a different order placed on each page. The various pages of the book were numbered and filled out as follows:

Cookbook for Division Problem

Page	*Contents of Page*
0	0 (Written in ink)
1	1 (Written in ink)
10	523 (Written in ink)
11	47 (Written in ink)
100	A blank page (For writing quotients)
101	A blank page (For writing remainders)

(The following are the orders that appear on the flow chart.)

110	*Write* the contents of Page 10 on Page 101 in pencil and turn to page 111.
111	*Write* the contents of page 0 on page 100 in pencil and turn to page 1000.
1000	*Compare* the contents of page 11 with the contents of page 101. If the contents of page 101 is smaller, turn to page 1011. If not, turn to page 1001.
1001	*Subtract* the contents of page 11 from the contents of page 101. *Place* your answer, in pencil, on page 101 after *erasing* what was there before. Turn to page 1010.
1010	*Add* the contents of page 1 to the contents of page 100. *Write* the answer, in pencil, on page 100 after *erasing* what was there before. Turn to page 1000.
1011	*Print* in ink on the back cover of your cookbook the contents of pages 100 and 101. Turn to page 1100.
1100	*Stop* work. You have solved the problem!

That is the complete cookbook. The students were instructed to write on the front cover, "Turn to page 110 and go ahead!"

The connection between the cookbook and the logical flow chart of the division example was made by coding the page numbers above the appropriate flowchart orders. The students were then turned loose on their cookbooks with pencils and erasers and asked to obey instructions. When they had finally emerged with the answer to the problem, they were told that they had been behaving as human digital computers, and that it remained only to show how each of the orders they obeyed could be carried out by electronic mechanisms.

This was done during the remainder of the term, but since there is plenty of literature available in the field, a brief summary of the meetings will suffice. The computer's basic components of input, storage or memory, arithmetic unit, output, and control were explained. A meeting was devoted to the subject of coding, at which point it became possible to explain how the computer has the remarkable flexibility of changing its own orders as it goes along, in keeping with a preassigned routine. Advantage was taken of the great publicity given to the computer and its effects on industry, commerce, and research. At the last meeting the great need for trained personnel in the computer field was stressed, and students were urged to consider this in making their plans for a career.

Enough interest was aroused to justify another term of club activity, and plans are afoot for the next semester. We hope to teach some elementary Boolean algebra, apply it to switching circuits, and finally to build some simple "electronic brains." We expect to take advantage of the "Geniacs" kit of Berkeley Associates that has been advertised in the pages of The Mathematics Teacher.[2]

We feel that the club program just described lies along the lines recommended by Northrop's article, "Mathematics and the Secondary School Curriculum."[3] It might be well, however, to underline the note of caution stressed in this article. There are gathering signs that a full-scale movement is getting under way to revamp the high school curriculum to include topics of importance in modern mathematics.

[2] *Geniacs Kit,* Berkeley Enterprises, Inc., 513 Avenue of the Americas, New York 11, New York. $17.95.

[3] E. P. Northrop, "Modern Mathematics and the Secondary School Curriculum," The Mathematics Teacher XLVIII (October 1955), 386.

In seeking these important benefits let us not lose sight of the very real values that are featured in our present course of study. In high school mathematics, the student learns the very alphabet of the subject. He learns how to express ideas in algebraic language, and the nature of postulational thinking. The logical hiatuses of the Euclidean system are more than made up for by their immense importance in the history of our culture and in the practical world about us. Let us be cautious before we scrap our traditional material in favor of ideas that are necessarily more sophisticated and more abstract. The results might not be what those who wish to reform our curriculum are asking for!

Have you read?

CULTER, ANN, "You Too Can Be a Mathematical Genius," *Esquire*, January 1957, pp. 58, 119–120.

5132437201×452736502785—this little multiplication problem was completed in just a little over one minute by a boy of less than average ability. How? By the Trachtenberg system of mathematics. You don't believe it? Well, try it.

This article gives you nearly all you need to know about the system devised by Jakow Trachtenberg while he was a prisoner in a German camp. What makes this system outstanding? It is complete, it eliminates all drudgery, it can be done by the mentally retarded, it only requires the ability to count to 10, it is easy to learn by young or old, and errors are easily found. What more could one hope for? Of course there are a few rules to learn, but your students will be fascinated even if the system does not remove all the difficulties in mathematics.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

FEHR, H. F., and SYER, H. W., "How Can Mathematics Promote International Understanding?" *The Bulletin of the National Association of Secondary School Principals*, December 1956, pp. 104–117.

Should the teaching of international understanding be left to the social sciences? The authors of this article think not. You will want to read their views; they make sense. For example:

Mathematics is a universal language as well as a body of content. What makes this so?

The values of mathematics are the same values required for better international understanding, such as deductive thinking, proper rigor, symbolic representation, relativeness, proper generalization, and permanency.

The International Congress of Mathematicians encompasses 40 nations. This face-to-face contact at meetings does much to promote understanding. For example, at the Netherlands' meeting in 1954, topics were discussed from highest mathematical subjects to what shall be provided for students of 16 to 21. Commissions were established to study secondary school mathematics, and the role of modern mathematics.

All this tends to bring peoples together. The article also gives suggestions for each school, such as exchange of mathematics problems, exchange of examinations, and others.

Mathematics can and should promote better international relations.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

SCHULT, VERYL, "Mathematics Today," *NEA Journal*, January, 1957, pp. 24–26.

For a quick symmary: what is today? what will probably be tomorrow? Read this short article for answers briefly given to questions similar to the following:

Are mathematics curriculums adjusted to today's needs?

Are gifted students being challenged?

Do instructors use what they know about how pupils learn?

Are colleges helping solve the problems?

Are students aware of the many opportunities in mathematics?

Miss Schult has packed this article full of ideas and questions. Read it.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Foundations of algebra[1]

BRUCE E. MESERVE, *Montclair Teachers College, Montclair, New Jersey.*

The structure and logic of modern algebra are a fundamental "piece of knowledge" for high school teachers of modern mathematics.

My topic—foundations of algebra—may look formidable to you; however we shall strive to stimulate your imagination by describing the "house of mathematics," especially that portion of the house and its foundations which may be associated with algebra.

The house of mathematics is certainly a mansion of many rooms. For some people, it is a stately Victorian mansion, appearing in excellent repair in a few of its parts, but gradually falling apart elsewhere. Many people still think of mathematics as having the ruggedness of a log cabin with cracks in its structure that encourage many doubting breezes and result in very little comfort for any but its hardiest inhabitants. Still other people think of mathematics as they do of a modern ranch style house with its single slab of solid foundation. You may pick the image that best fits your concept of mathematics.

For me, mathematics is like my own house—old by modern standards, but very utilitarian, amenable to modern innovations, and much more sturdily built than many modern houses. There is room to move around comfortably, and room for each member of the family to develop his own special interests while maintaining a decent respect for the needs and the requirements of others. Primarily there is a solid foundation and structure (much of which I am only beginning to discover) which offers great promise of meeting our needs if developed with good taste, considerable imagination, and with an eye to the future. I am sure that the process of remodeling my house to meet the needs of the family will be a continuous and never-ending process. I am also sure that in the house of mathematics there will be and must be a continuous remodeling in the sense of redecorating the various rooms (changing our point of view in various areas), introducing some of the modern conveniences, and occasionally making a complete reorganization of a room or even adding rooms.

Just as, a few years ago, the lighting of many houses was converted from candles and oil lamps to gas lamps, and later to electric lights, so the house of mathematics must occasionally undergo noticeable changes to maintain its service to mankind and its position of leadership in setting the styles in our scientific culture.

In a very real sense, the house of mathematics faces a drastic remodeling as machine computation is developed and becomes increasingly useful in the handling of data. Just as the introduction of electric lights in our houses provided an increase in the utility of each of our rooms, so the introduction of machine computation must be expected to provide an increase in the utility of each of the rooms in the house of mathematics. In some areas its influence will be tremendous; in other areas it will serve primarily as a new tool for increasing the effectiveness of that aspect of mathematics. As the new innova-

[1] Presented April 12, 1956 at the Thirty-fourth Annual Meeting of the National Council of Teachers of Mathematics, Milwaukee, Wisconsin.

tion is developed in the years to come, it will undoubtedly be desirable for us to reorganize our thinking; such innovations will lead to a better understanding of mathematics. However, as we introduce new ideas and procedures in order to understand mathematics better, we also introduce changes in mathematics. The remodeled houses are built on the same foundations, and serve their intended purposes better than before, but they are not exactly the same after being remodeled.

Turning specifically to algebra, we can say briefly that algebra is based upon:

Numbers associated with sets of elements,
Number symbols,
Variables,
Order relations (more, less) among numbers,
Equations indicating that two symbols represent the same number,
The operations of addition and multiplication,
Their inverse operations of subtraction and division,
The associative, commutative, and distributive laws,
Deductions from these and other laws,
The concepts of group, ring, and field.

The importance of these aspects of algebra lies in their numerous interpretations and their use with many different-appearing sets of elements. There is an algebra of logic that is based upon the theory of sets and is very useful in handling composite statements. The effectiveness of Boolean algebras in the study of electrical circuits is now recognized. An algebra of vectors is advantageous in studying forces and velocities. These and other algebraic systems designed to provide mathematical interpretations of various aspects of the physical universe illustrate the tremendous effectiveness of algebra and the need for solid foundations for such a multiplicity of applications. However, it is very difficult to keep a discussion of these aspects of algebra within the range of our usual experiences and our common mathematical vocabulary. Thus it would be well to return to the analogy of the house of mathematics to consider further the foundations of algebra.

Recall that the house of mathematics is due for extensive remodeling because of the introduction of machine computation. To be prepared for this remodeling and to shape the changes in line with our needs, we should reappraise the structure of mathematics in terms of our present and anticipated knowledge and needs. Here again our analogy between the house of mathematics and the physical houses that we live in is most useful. We must reappraise the house of mathematics from a twentieth-century point of view. In remodeling your home, you would not make a decision regarding the desirability of a change simply by comparing it with the cave of a primitive man, an igloo, a skin house prevalent in an undeveloped culture, the log cabin common a few centuries ago, or even the best of nineteenth-century houses. You would think in terms of running water, inside toilets, electric lights, thermostatically controlled ovens, radios, and television. You would think of inner spring mattresses instead of bins of straw or mats of animal skins. You might even look forward to the day when television pictures will be projected on the walls of your living room instead of being confined to small tubes.

Likewise, you should not make a decision regarding the desirability of a change in mathematics simply by comparing your present concept of mathematics with the elementary number concepts of primitive men, the geometric algebra and the formal organization of the geometry of the Greeks two thousand years ago, or even the analytic geometry and calculus of the eighteenth and nineteenth centuries. You should not think of computations in terms of the finger reckoning of early men, the abacus, the hand-operated adding machine, or even

the desk calculator. You should be thinking of many devices for computations—devices selected according to the complexity of the calculation and including modern digital and analogue computers for the most difficult problems. You should be thinking of modern conveniences in mathematics as you do of modern conveniences in your homes—conveniences which enhance not only the beauty but also the utility of the structure, conveniences which bring power into the house—power for labor-saving devices, for improving our knowledge, and for increasing our appreciation of the house.

In the house of mathematics, as in other houses, the modern conveniences are outgrowths of the lesser conveniences of previous centuries. In the twentieth century we should take special recognition of sets of elements, symbols (often called variables) which may be replaced by any element of a specified set, relationships among elements, functions, operations upon elements, and the reasoning that underlies our mathematical structure—inductive and deductive proofs, the necessity of having undefined terms and unproved assumptions, the method of obtaining assumptions by abstracting properties from our physical environment, and by generalizing previously accepted assumptions. Finally, we should be looking at our house of mathematics as an independent structure possessing an inner unity and serving us effectively because of the reliability of its representation of the quantitative aspects and regular patterns of the world around us.

When primitive men left their caves to make their own crude huts, they recognized certain properties of the caves that were essential to their use as homes and they incorporated these properties into their houses. Likewise the house of mathematics has been developed by abstracting properties of the space in which we live.

Our concept of number is based upon one-to-one correspondences between sets of physical objects such as three fingers and three fish or three pebbles and three sheep. Gradually the common property of sets that could be matched in this way was recognized as a number property and extended (generalized) through various stages that we may now identify with positive integers, positive rational numbers, signed rational numbers, real numbers, complex numbers, and other abstract number systems.

At first the necessary properties of numbers could be based solely upon the properties of sets of physical objects associated with the numbers. As man extended his concept of number, this reliance upon sets of physical objects became awkward and rules were devised for combining numbers and for expressing relationships among numbers. In our secondary schools today we need to take our students through this step of extending their concept of numbers from properties of sets of physical objects to properties of symbols depending upon rules. We need to build up in our students an appreciation for the structure of the house of mathematics as a well-built unit on a firm foundation, rather than as a decrepit ark about to fall down under the weight of its own propositions and theorems of questionable utility.

The keystone in the foundation of algebra is arithmetic—arithmetic with its sets of numbers and its rules for combining those numbers. On the one side of the keystone we have the numbers of arithmetic, on the other side, the symbols of algebra that may be replaced by numbers of arithmetic. Above the arch we have the structure of algebra depending upon both the numbers and the properties of arithmetic and the generalization of these numbers and properties for other sets of symbols and operations upon the symbols. As we enter the house of mathematics through the archway leading to the areas associated with arithmetic and algebra, we must recognize the dependence of all of our future work upon this arch. Like the two pillars that we are told Samson pulled down in destroying an ancient

temple, the numbers of arithmetic and the symbols of algebra, together with the rules for using each, provide the main pillars for the structure of mathematics. In the words of Dr. Howard Bevis, president of Ohio State University and recently appointed by President Eisenhower to head a National Committee for the Development of Scientists and Engineers, "More students must learn arithmetic."

Man has not always appreciated the importance of the rules (axioms) of arithmetic and algebra. Indeed, there is a real question as to whether we are sufficiently cognizant of them today in our schools and in our elementary college courses.

With relation to secondary schools, I am thinking particularly of the equation principles and the definitions associated with the student's expanding concept of number as he encounters positive rational numbers, all rational numbers, real numbers, and, near the end of his high school career, the complex numbers. Deductive thinking is involved here just as much as in plane geometry. Indeed, many people feel that the deductions in algebra are much easier for the student than those in geometry. Unless we teachers, our administrative superiors, and our textbook writers recognize the importance of these foundations of arithmetic and algebra, there actually is a danger that, for our society, the house of mathematics will noticeably sag, dragging down with it many aspects of related sciences and leaving the remaining theoretical structure as a basis for the achievements of a more vigorous culture.

This remark is not an idle threat in our present world with its war tension and race for nuclear supremacy. Rather, consider it a calculated remark with an element of truth dependent upon our vigor in adopting a twentieth-century mathematical structure rather than placing reliance solely upon cave-man mathematics, the mathematics of a Greek culture two thousand years ago, or even the mathematics of a European culture two centuries ago. Any realistic person must recognize that if a twentieth-century point of view appears in a secondary school class in the United States today, it is considered either highly incidental and irregular or out of place. At the same time, we must recognize that we cannot teach the breadth of twentieth-century mathematics in our secondary schools. If we wish our culture to survive, our task is to prepare our students for twentieth-century mathematics and to lay the foundations for a vigorous scientific culture in this part of the world.

How can we properly prepare our students? How can we lay the proper foundations? Obviously a complete answer to these questions would require much more than the few pages allocated to this article, and their specification would be beyond the capacity of any one individual. However, this does not mean that we should do nothing.

Each of us should accept the responsibility of fortifying his own thinking through continuous study and increasing his awareness of the problems before us. The house of mathematics, and indeed the future of the society in which we live, depends upon the way in which we, collectively, accept this responsibility.

What can we study? Where can we look for further guidance and inspiration in this matter? Several individuals have written books endeavoring to provide a proper background for our work. But this is not a time for glorifying individuals. Our best hope lies in the National Council of Teachers of Mathematics and its publications. The yearbook entitled *Insights into Modern Mathematics* will be available by our 1957 annual meeting. This yearbook contains eleven chapters in which prominent mathematicians endeavor to describe the foundations and the elementary concepts of various aspects of mathematics. The first four of these chapters—"The Concept of Number" by Ivan Niven, "Operating with Sets" by E. J. McShane, "Deductive Methods in Mathematics" by Carl B. Allendoerfer, and "Algebra" by Saunders

MacLane—should provide excellent reading for anyone who wishes to understand the foundations of algebra. The concluding chapter of the yearbook is devoted to a discussion of the implications for the mathematics curriculum. It is hoped that this chapter will improve the usefulness of the yearbook as you seek to interpret the earlier chapters in the light of your daily work. Other yearbooks (past and future) should also be helpful.

What can you read during the next year? Study the articles in THE MATHEMATICS TEACHER and *The Arithmetic Teacher*. Subscribe to these periodicals if you have not already done so. If you find an article that appears helpful, see if its author has written other articles or published books that might be of value to you. See if he has written textbooks at the level at which you are teaching. If so, look in those texts to see whether they reflect his present thinking. However, in this last regard it is essential that you recognize the author's difficult position. He can not stress modern concepts in a textbook until those concepts have received at least moderate general acceptance. But he can try to prepare students and teachers for those concepts.

In conclusion let me return to the house of mathematics—this house of many rooms in various states of repair and renovation. The house of mathematics has a foundation that is becoming secure from a logical point of view. From a practical point of view the foundations, especially those in arithmetic and algebra, need remodeling in most of our classrooms. This remodeling is needed to prepare individuals to participate in the further remodeling of the superstructure of the house and to prepare its inhabitants to appreciate and profit by living in the house of mathematics. Thus our primary concern is that mathematics be given an opportunity to serve our society as effectively as possible. We do not need to worry about the structure of mathematics. We do need to worry about the structure of our society in its dependence upon mathematics.

The house of mathematics provides a vantage point for seeing and understanding the quantitative aspects of our physical universe. The views of the physical universe vary as we move from room to room—from arithmetic to algebra, to analysis, to statistics, and to other branches of mathematics. Our understanding of the universe increases as we are able to appreciate the different points of view. For the purposes of this paper we must recognize that the numbers and rules of arithmetic provide the key to our understanding algebra, and, together with algebra, provide the most effective basis we have for entering the house of mathematics.

It is a profoundly erroneous truism, repeated in all copy books and by eminent people when they are making speeches, that we should cultivate the habit of thinking of what we are doing. The precise opposite is the case. Civilization advances by extending the number of important operations which we perform without thinking about them.—*Alfred North Whitehead.*

Curiosity and culture[1]

F. LYNWOOD WREN, *George Peabody College for Teachers, Nashville, Tennessee.*

From the faith that nature consists of a harmonious structure of mathematical laws and from an insatiable curiosity man has learned to build mechanical models which are flexible enough to structure events in the physical world.

PEOPLE ARE CURIOUS. They are innately desirous to see what is novel or to discover what is unknown. They possess an inquiring disposition which constantly urges them to make an effort to obtain information by questioning others and to seek into the facts, causes, or principles which control their work and play, their earning and spending, their eating and sleeping, their coming and going, their joy and sorrow. This inborn, impelling inquisitiveness manifests itself in many different forms, ranging in maturity from that of the child who exclaims in glee:

> Oh see the pretty clock!
> What makes it go tick-tock?

to that of the philosopher who, with Socrates, wonders

> . . . whether all this which they call the universe is left to the guidance of unreason and chance medley, or, on the contrary, as our fathers have declared, is ordered and governed by a marvelous intelligence and wisdom.

Curiosity, or the disposition to inquire, "is one of the permanent and certain characteristics of a vigorous mind." The ebullient spontaneity with which it functions in the mind of man was effectively epitomized by the six-year-old who, when abruptly startled by a strange mixture of unexplainable noises, asked, "Daddy, why did that man say that something to that what?" Through the ages man and child alike, each by ways and means peculiar to his own individual mental stature, have sought to discover the why and how of what happened where and when.

The chronicle of man's effort to seek out the unknown and to compound or decompose the known in the laboratory of intellectual insight charts the course of our evolving culture. Some efforts are fruitless and become lost immediately in the abyss of obscurity; others, though they dazzle by a flash of superficial brilliance as they point up temporary issues, leave no lasting effects. From the crucible of time alone emerge those efforts which perpetuate themselves in sublime glory as phoenixlike they revive from the ashes of ill-remembered significance with quincentennial persistence, or as the daystars sink

> . . . in the ocean's bed,
> And yet anon repair their drooping head,
> And trick their beams, and with new-spangled ore
> Flame in the forehead of the morning sky.

Innocent and unheralded may be the circumstance or event which sounds the challenge to man's curiosity and entices him into the realm of immortal accomplishment. A Greek philosopher and his interest in the mystery of number; a Polish justice of the peace and a German ministerial student and their astronomical hobbies; an Italian teen-ager and his concern

[1] An address delivered at the Sixteenth Summer Meeting of the National Council of Teachers of Mathematics. Published with permission of the Bureau of Publications, George Peabody College for Teachers, Nashville, Tennessee.

over a large hanging lamp; an English farmer boy and his observance of a falling apple; a French philosopher and his search for the truth; an international quintet and their refusal to accept the absolute; a German refugee and his curious suspicion of the obvious: fortunate unions of curious minds and unexplained phenomena whose offspring, individually and collectively, have left their indelible imprint upon the cumulation of experience which has shaped the pattern of the intellectual content of our civilization, the fabric of our culture.

The Greek philosopher and mathematician, Pythagoras, is somewhat of a question mark on the pages of early Greek history. There are very few educated people, however, for whom the title, *Pythagorean Theorem*, is not associated with a proposition which states a very fundamental relation between the sides of a right triangle. In spite of this fact there seems to be strong evidence that this property of right triangles was known and used by the Orientals thousands of years before the time of Pythagoras. True, their knowledge was based on intuitive insight or empirical evidence alone, but history is also now saying that there is serious doubt that Pythagoras was temperamentally disposed to be interested in the establishment of the logical validity of this theorem or any other theorem. Regardless of these doubts of latter days, the earlier story was the inspiration of a poetic gem from the German language. Although much of the subtle humor and metric nicety of the poem become lost among the inadequacies of translation, the unknown translator was able to catch the significant meassage of the poet. Earlier historians had it that Pythagoras became so jubilant over the discovery of this famous theorem that he sacrificed one hundred head of oxen. This story and the fact that the German word for "oxen" also carries a vernacular significance of "blockhead" enabled the German poet Adelbert von Chamisso to engage in a rather clever bit of metaphorical implication.

On the Pythagorean Theorem

Truth lasts throughout eternity,
When once the stupid world its light discerns,
The theorem, coupled with Pythagoras' name,
Holds true today, as't did in olden times.

A splendid sacrifice Pythagoras brought
The gods, who blessed him with this ray divine;
A great burnt offering of a hundred kine [*Ochsen*]
Proclaimed afar the sage's gratitude.

Now since that day, all blockheads [*Ochsen*] when they scent
New truth about to see the light of day,
In frightful bellowings manifest their dismay;

Pythagoras fills them all with terror;
And powerless to shut out light by error,
In sheer despair they shut their eyes and tremble.

Regardless of what history may say, now or eventually, concerning Pythagoras' relation to this significant theorem from geometry or to other mathematical or philosophical speculations, it is known that he became the titular head of a semi-religious cult "which persisted for many centuries and exerted a tremendous influence on scientific and religious thinking." The theme which seemed to inspire and control the intellectual activities of the Pythagoreans was: *Number rules the universe*. It is a bit speculative whether by this dictum they might have been displaying keen scientific precognition or merely exhibiting a sort of supernumerological concept of the scheme of creation. Whether they discovered the right triangle theorem or not, they did recognize it as a basic law of nature. It led them to discover that numbers such as $\sqrt{2}$ have a peculiar significance, though the full implications of the irrationality of these numbers were not clearly established until the nineteenth century. Could it be that their recognition of the numerical properties of the right triangle, which we so simply state in the formula

$$c^2=a^2+b^2,$$

was but an indication that by their dictum "Number rules the universe," they were foreshadowing the formularization of all the laws of nature?

Their conviction that the foundation of true philosophy lay in numbers and their relations led the Pythagoreans to the observation "that musical strings of equal lengths stretched by weights having the proportions of 1:2, 2:3, 3:4, produced intervals which were an octave, a fifth, and a fourth." Thus harmonious combinations of plucked strings can be expressed as ratios of whole numbers. This discovery, coupled with the fact that they believed that bodies moving in space would produce harmonious sound, led them to draw a parallel between the seven known intervals in the musical scale and the seven known planets crossing the heavens. This "harmony of the spheres" thus reduced the motion of the planets to no more than number relations.

If not *the* earliest, the Pythagoreans were among the earliest of numerologists. The natural numbers, the numbers with which we count, not only had scientific significance in the philosophy of the Pythagoreans, they also had many mystic implications. I shall mention but one such peculiar affinity which they attached to number, and it is mentioned merely for your speculation. I would certainly hesitate to develop the theme. They regarded all even numbers as feminine and all odd numbers as male. Lest someone beat me to the observation, may I state parenthetically here that very likely there are quite a few "female members," in fact there are probably some within this audience, who, while they may not accept the Pythagorean characterization of odd numbers as male, they, with no equivocation but with rather firm conviction, do characterize male numbers as odd. Now an even, or female, number can be broken down into the sum of two even numbers; for example $8=4+4$, $4=2+2$, and so on. Ultimately, however this breakdown is brought to a stop by the fact that $2=1+1$, the sum of two odd, or male, numbers. Thus the male of the species keeps the female of the species from going all to pieces. Furthermore, such observations led them, not me, to the speculation that male numbers are appropriate to constancy and knowledge while the female numbers can express themselves only through inconstant opinion.

This strange mixture of the scientific, pseudoscientific, and occult, known to us as the philosophy of the Pythagoreans, has proved to be the fountainhead of many ideas left floating on the stream of thought. Over the ages, as this stream has carved its course through the canyons of unyielding doubt, the forests of confused superstition, and the plains of imaginative intutition, it has deposited ideas and thoughts along the way, some of which have wilted under the heat of critical inquiry while others have germinated, and, nourished by the cooling waters of inspiration and insight, have borne the fruit of the culture of our civilization.

The Polish-born Nicolaus Copernicus, while serving as a justice of the peace and steward of the church properties of the Cathedral of Frauenburg, spent a great deal of his time in one of the small towers of the cathedral observing and studying the motion of the planets. Though his instruments and techniques were crude, his inspiration and insight were keen. He had become imbued with the spirit of the Pythagorean doctrine that "Number rules the universe" and that nature was a harmonious structure of mathematical laws. Furthermore, he became convinced that the key to the comprehension of this secret harmony lay in the heliocentric theory of the solar system, as previously anticipated by Aristarchus, and not in the then accepted geocentric theory of Claudius Ptolemy.

Through his untiring efforts, Copernicus was successful in arriving at a mathematical account of celestial motion which was much simpler than the many epicycles and eccentrics of the Ptolemaic system. He dared to attempt to think through what others had called absurd and, after thirty years of devoted effort, was able to proclaim:

> We find, therefore, under this orderly arrangement, a wonderful symmetry in the universe, and a definite relation of harmony in the motion and magnitude of the orbs, of a kind that is not possible in any other way.

Though Copernicus greatly simplified the study of the behavior of the planets, he was not able to free himself from the shackles of their assumed circular motion. It remained for a young German ministerial student to have the vision and imagination sufficient to call into practical service the ellipse, a curve theoretically developed by the Greeks nearly two thousand years earlier. Johann Kepler, while studying for the ministry, became so interested in the work of Copernicus that, somewhat as a hobby, he sought the aid of private instruction in the effort to understand better his proposed theory. The challenge and interest which Kepler found in this study soon resulted in his hobby becoming his chosen profession. The most famous results that came from his inspired efforts are the three laws of planetary motion which bear Kepler's name and which, to this day, continue to carry great scientific significance.

It is very fortunate indeed that Copernicus and Kepler were not "practical" men, interested primarily in the investigation of that which might be of immediate use. Their motivation was merely an intense and sincere desire to seek out a simple theory to explain the motion of the planets. For them the belief in a number-ruled universe was "wrapped in the silver tissue of conviction." Therein lay the source of their somewhat self-centered assurance that "if the theory did not fit all the facts, it was too bad for the facts." From their combined efforts, purely theoretical though they were, our modern culture enjoys the practical heritage of a simplified rationalization of the uniformity and invariability of nature.

Many such instances from the fields of mathematics and science emphasize the fact that we should, at all times, be alert lest the comfortable simplicity and the somewhat pseudorealistic assurance of pragmatism cause us, in our educational innocence, to forget that theoretical thinking can, and does, lead us to many significant results which experience cannot possibly foresee. Not always is it true that close attention to the immediately useful is the most effective way of being practical. Furthermore, we should remember that we have made a miserable forfeit in our debt to future generations when we, deliberately or unintentionally, permit the pattern of our educational programs to be molded in the faulty furnace of a philosophy which subdues the durable values of culture and discipline to an overpowering white heat of temporal utility.

Daydreaming in church is not so likely to persuade one to pursue a line of concentrated constructive thought as it is to dissuade one from engaging in such an endeavor. Not so for Galileo Galilei! While a teen-age medical student at the University of Pisa he engaged in a bit of church daydreaming out of which eventually evolved the modern concept of scientific activity. While not as attentive as he might have been to the church service, his curiosity was aroused by the swinging back and forth of a large hanging lamp. Using his pulse beat to measure the time of its swing, he was led to the significant discovery that the time required by a pendulum to complete its swing is independent of whether the swing is through a wide arc or a narrow arc—a simple observation but an astounding discovery that was destined to leave its imprint on the culture of future generations. This is but one of many instances of Galileo's indulgence of his curiosity, the cumulative effect of which was to lead him to his revolutionary concept of scientific activity, namely, to obtain "quantitative descriptions of scientific phenomena independently of any physical explanations."

Scientists prior to Galileo's time were interested primarily in "why" the phenomena of nature happen as they do. Galileo became interested in "how" they

happen. Through the mathematization of such phenomena as distance, time, velocity, and acceleration he sought to determine the relations that exist between them. The formula became a powerful instrument in his effort to weld "experiment and mathematics into a single implement of discovery and exploration." Out of this quantification of natural phenomena evolved a pattern of scientific endeavor that has exercised a great influence in the shaping of our modern civilization.

The contemporary efforts of Kepler and Galileo rewarded each with success: for the former it was the quantification of celestial motion and for the latter it was the quantification of terrestrial motion. The similarities in the two distinct theories gave inspiration and the dissimilarities sounded challenge to the farm boy, Isaac Newton, to seek a single universal law that would describe motion, whether celestial or terrestrial. Legend has it that a falling apple spurred his curiosity to search for that unifying principle which would identify the earth's pull on objects with the sun's attraction on planets. In Lord Byron's words:

When Newton saw an apple fall, he found
In that slight startle from his contemplation—
'Tis said (for I'll not answer above ground
For any sage's creed or calculation)—

A mode of proving that the earth turn'd round
In a most natural whirl, called 'gravitation';
And this is the sole mortal who could grapple,
Since Adam, with a fall, or with an apple.

Newton acquired from Galileo the basic facts which he restated in the form we now recognize as the first two of Newton's three laws of motion:

I. *Every body will continue in its state of rest or of uniform motion in a straight line except in so far as it is compelled to change that state by impressed force.*

II. *Rate of change of momentum is proportional to the impressed force and takes place in the line in which the force acts.*

The fall of the apple seemed to be in perfect accord with the principles implied by the first law. Furthermore, by this time, general scientific acceptance had been accorded to Kepler's first law of planetary motion, namely: "*The planets move around the Sun in ellipses; the Sun is at one focus of these ellipses.*" The inquisitive mind of the curious Newton sought the underlying principle which would relate these twins of attraction, apple-to-earth and planet-to-sun. He devoted twenty years of his life to persistent search for the answer to such questions as: "What can explain the motion of both the apple and the planet?" "Why do the planets not move in straight lines?" Strangely enough the key to the answer to these questions is to be found, at least according to some historians, in the first three words of Newton's second law of motion, "rate of change."

To obtain a workable mathematical method for investigating rates of change, Newton was led to the discovery of his "Method of Fluxions." Subsequent analysis of the behavior of fluxions (today we call it differential calculus) led him to properties and techniques of the integral calculus. Prior to this discovery Newton had been able to take care of the behavior of mass particles, but what happened when these particles became the component parts of solid homogeneous spheres? How could he determine the resultant attraction of all these separate attractions, infinite in number? By methods of the integral calculus he was able to establish the fact that the combined effect of this myriad of attractions was the same as if the entire mass of the sphere were concentrated at a single point, the center of the sphere. When Newton had finally found the answer to this stubborn question, he was able to state the great unifying principle which we know today as his law of universal gravitation: *Any two particles of matter in the universe attract one another with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.* Once having stated this law, Newton was able to demonstrate

that Kepler's three laws of planetary motion could be derived from it. A momentous and gigantic step had been taken toward demonstrating the prophetic truth of the Pythagorean dictum, "Number rules the universe."

In a somewhat incidental way, mention has been made of Newton's magnificent contribution of the principles and techniques of calculus to the subject content of mathematics. The importance of this joint discovery with von Leibniz cannot be overemphasized, but it would take us too far from our theme to attempt to develop its significance. Suffice it to say that calculus has become the cornerstone of *analysis*, one of the vastest and most significant areas of modern mathematics.

To attempt to elaborate upon, or indeed merely to catalog, Newton's contributions to the fields of optics, celestial mechanics, dynamics, and mathematics would be a gargantuan task. His illustrious contemporary von Leibniz said: "Taking mathematics from the beginning of the world to the time when Newton lived, what he did was much the better half." The climax of all his endeavor was reached in the year 1687 when the *Philosophiae Naturalis Principia Mathematica* was published. The years 1684 to 1686, when he was engaged in the preparation of this momentous work, have been said to "mark one of the great epochs in the history of all human thought." Not only have the three books of this colossus of scientific endeavor been of great significance to the fields of science and mathematics, but the implications of the third book, the great treatise on the "System of the World," have left their impact on the fields of logic, philosophy, religion, literature, and the arts. They inspired efforts further to mathematize science; to emphasize the deductive pattern of reasoning; to establish a universal technical language; to standardize the meanings of words through the structure of dictionaries; "to reduce art to a system of rules and beauty to a series of characterizing formulas"; and to mathematize, for ready solution, all problems of mankind. A committee of the Fellows of the Royal Society recommended to the members that they use a "close, naked, natural way of speaking; positive expressions, clear senses, a native easiness; bringing all things as near the mathematical plainness as they [could]." Churchmen thought they recognized in this new emphasis on the quantification of natural phenomena and their controlling laws a threat to religion proper and to concepts such as those of God and the soul. It would be futile in such a short space to attempt an evaluation of the effect of the Newtonian influence upon our modern culture. We might, however, attempt a summarization by simply saying that the impact of the influence of the work of Newton and his contemporaries has been to give to the modern age a culture characterized by rationalism as opposed to the ecclesiasticism and feudalism of the cultures of earlier years.

Newton is credited with having at one time said, "If I have seen farther than Descartes, it is by standing on the shoulders of giants." Indeed, not the least among those intellectual giants, on whose shoulders Newton stood, was the French philosopher-mathematician René Descartes himself. It has been said of him that "he was the first influential thinker to demonstrate to the world the nature and value of mathematical method in man's search for truth." From his early childhood he had persisted in accepting nothing on mere authority; a habit which early led to the formulation of the question which was to be the inspiration of his life's work: What is truth, and what method should man use in his search for truth?

If a one-word description of Descartes' method were to be given, that word would be "doubt." He doubted everything that he might know with certainty. Throughout his life, which he spent in travel, soldiering, and reflection, he was continually seeking by what means he could

refute his doubt and replace it by certain knowledge. The answer came to him in the form of a dream during the night of November 10, 1619. From this dream he received the inspiration for his "universal mathematics," whose core was the application of the techniques and principles of algebra to the analysis of geometric properties. Today we call it Cartesian Geometry, Co-ordinate Geometry, or, more familiarly, Analytic Geometry. John Stuart Mill said "... [Analytic Geometry] far more than any of his metaphysical speculations, immortalized the name of Descartes, and constitutes the greatest single step ever made in the progress of the exact sciences."

To Descartes, mathematics appeared to be the only hope for exact knowledge. It brought forth logical reasons to support what it affirmed and therefore gave intellectual certainty. In his *Discours de la Méthode* he declared:

> I was especially pleased with mathematics, on account of the certitude and the evidence of its reasons; but I did not notice at all its true usage, and considering only the mechanical arts, I was surprised that foundations so strong and solid had not been used for larger buildings.

In his search through the sciences for method he found only two areas of thought, arithmetic and geometry, which he felt did not depend upon randon conjecture. For this reason they were chosen to furnish the criteria for the certainty of all knowledge. By finding out how it was that arithmetic and geometry were so successful in reaching truths that were certain, he thought that he would be able to discover certainty in other sciences. Thus he stated his guiding principle in these words: "... in our search for the direct road towards truth we should busy ourselves with no object about which we cannot attain a certitude equal to that of the demonstrations of Arithmetic and Geometry." This became to him a sort of open sesame in his search for the absolute, or that "which contains within itself the pure and simple essence of which we are in quest."

Descartes made many original and fresh contributions to the field of philosophy, and many call him the founder of modern philosophy. In the field of mathematics he, in a very real sense, remade geometry by removing the ban of Greek classicism and making modern geometry possible. With Adolphe Franck we might well say that

> The birth and development of Cartesianism represent ... one of the epochs of the intellectual history of the human being to which none other, since the most brilliant days of Greek philosophy, is comparable. It is nothing less than the birth and development of the modern spirit.

Descartes sought the absolute in whatever subject he chose to investigate just as he sought the dominant element in every mathematical inquiry. But what is this "absolute" for which he so earnestly sought? In the *Discours* there are many illustrations. One, in which the "absolute," as portrayed by a geometric theorem, is used to give support to the argument for the "absolute" of a Perfect Being:

> ... I saw very well that if we suppose a triangle to be given, the three angles must certainly be equal to two right angles; but for all that, I saw no reason to be assured that there was any such triangle in existence, while on the contrary, upon returning to the examination of the idea which I had of a Perfect Being, I found that in this case existence was implied in the same manner in which the equality of its three angles to two right angles is implied in the idea of a triangle; ... and that, consequently, it is at least as certain that God who is this Perfect Being, is or exists, as any demonstration of geometry can possibly be.

While Descartes was a nonconformist in many respects, he did conform with the pattern of the previous millennia of human experience in accepting the Euclidean system of geometrical thought. To him, and to others through the ages, Euclid was thought "to have discovered an absolute truth or a necessary mode of human perception in his geometry."

Second only to the Holy Bible in perennial popularity has been the *Elements*, a collection of all the mathematical works of Euclid's contemporaries and predecessors. In the geometrical portion of this cele-

brated work, Euclid attempted to present all the then-known geometrical subject matter in such a logical array that all statements would find their ultimate reality in a set of fundamental assumptions. The inherent order of this deductive exposition became the dominant element in geometry which made such an impression upon Descartes that he made it an integral part of the basic method of his philosophy.

To the philosopher a basic set of assumptions, other than being simple and categorical, should be consistent and independent. As he gathers from the impressions of experience, or from whatever source he may choose, those principles from which his deductions are to follow, he wants them such that they will not lead to any inconsistent conclusions. Furthermore, while not absolutely necessary, it is logically esthetic that this list of assumptions be reduced to an absolute minimum in order that a maximum number of "truths" will be derived through careful reasoning.

Despite the distinction of synonymy with truth, ascribed to Euclid's geometry, there were two of his postulates which disturbed the minds of thinkers. Euclid himself seemed to have had some reservations concerning them. They were:

Postulate II. *A line segment may be extended indefinitely in either direction.*

Postulate V. *Through a given point not on a given line there can pass one and only one line parallel to the given line.*

Innocent statements, yet are they true beyond the horizons of experience, or merely obvious within the observable environment? In fact, by looking into the distance along a railroad track, one can even within the domain of observation very easily be led to misgivings concerning the parallel postulate. Another disturbing question arose. Are they derivable from the other postulates? The fifth, or parallel postulate, provoked the greater concern. Many efforts were made to prove by direct argumentation that it could be derived from the remaining postulates. Other efforts attempted in the same manner to prove that it could not.

After two thousand years of such fruitless efforts, Saccheri, an Italian Jesuit priest, attacked the problem by the indirect approach. He suggested the substitution of either of two possible contradictions to the parallel postulate, keeping all other postulates intact, and then investigating the possibility or impossibility of inconsistencies arising. He was on the threshold of mathematical immortality, but he let the shackles of tradition prevent him from stepping across. He seemed obsessed with the idea that Euclid's geometry was perfect. Although all his efforts led him to the contrary conclusion, he was able to construct satisfactory explanations and finally, in 1733, published his book, *Euclid Vindicated from All Defects*. A modern, more familiar version of the same type of argument is *He Bought the New-Model Car Regardless.*

Twenty centuries of barren effort had convinced many geometers that the final settlement of the theory of parallels involved a problem that was not solvable. In fact, one school of thought, by official announcement, had reaffirmed the Kantian edict that there can be no other geometry than Euclid's. But three men, the German Gauss, the Russian Lobachevsky, and the Hungarian Bolyai, following in the footsteps of Saccheri, independently and somewhat contemporaneously dared contradict Postulate V and stand by the consequence of the deductions therefrom. Gauss did not have the moral courage of the other two and failed to announce to the world, as they did, that there *can* be other geometries as valid as Euclid's. This tradition-shattering discovery was published by Lobachevsky in 1830 and Bolyai in 1832.

Language complications and inadequate publicity resulted in virtual contemporaneity of the two publications, at least as far as the Western world was concerned. In 1854 another German, Bern-

hard Riemann, dared contradict the assertion of Postulate II which, in turn, led him to still another contradicting substitute for Postulate V. He was able to establish still another geometry just as valid as Euclid's or as Lobachevsky's.

Since these non-Euclidean geometries retain intact all Euclidean postulates other than those contradicted, it follows that those theorems which do not depend on the contradicted postulates remain just as in the Euclidean geometry. Where the parallel postulate is involved there are, of course, theorems which contradict the corresponding Euclidean theorems.

Among the most important of these contradictions are the three versions of the theorem which deals with the sum of the angles of a triangle. Descartes found assurance in the Euclidean assertion that the sum of the angles of a triangle is two right angles. Lobachevsky was able to demonstrate that this same angle sum is *less than* two right angles, while Riemann, just as logically, was able to establish that it is *greater than* two right angles. Could that distant rumbling be from Descartes' restlessness in his grave?

Whether we accept the Euclidean statement concerning the sum of the angles of a triangle or one of the two non-Euclidean statements is not subject to our choice. What is subject to our arbitrary choice is to decide which of the three principles of parallelism we wish to accept as basic. Once we have made this choice, we must accept the relationship to which it leads through sound logical deduction. As to agreement with experience, the Euclidean sum checks with what we find in the narrow confines of our environment which conforms to existence on the level surface of a plane. Thus, for civil engineers and contractors as they survey our land, build our highways, and erect our buildings, the sum of the angles of any triangle with which they deal will always be two right angles. The Riemannian sum checks with what pilots of our planes and ships find as their experience is extended to the spherical surface of our earth. Just as the plane surface conforms to the postulates of Euclidean geometry, so the surface of a sphere conforms to the postulates of Riemannian geometry. Thus, for the navigators of our ships, whether they fly by air or sail by sea, the sum of the angles of many of their triangles is greater than two right angles. Who is there to say that there may not be in outer space awaiting man's discovery some pseudosphere on which Lobachevsky's sum holds true?

This international quintet, Saccheri, Gauss, Lobachevsky, Bolyai, and Riemann, dared challenge that which had been accepted as absolute by a prejudice sanctified by two thousand years. This was a cataclysmic event in the history of thought because, by so doing, they freed the mind of man to reject the evidence of the senses for the sake of what the mind might produce. When weighed in the scales of creative thought and significant culture, what greater freedom can there be? Such freedom has given us today a culture in which, more surely than in that of our ancestors, the conventional is no longer inviolable, the absurd may well become respectable, the absolute is subject to dispute, and the obvious has become suspect.

To distinguish absolute motion from relative motion, Newton conceived of space as a physical reality, stationary and immovable, and thus satisfactory as a fixed frame of reference for the motion of the stars. This point of view with the accompanying ether theory to provide the medium needed for the transmission of light waves prevailed for approximately two centuries.

The classic Michelson-Morley light experiment in 1881 confronted scientists with a terrific, thought-provoking, soul-searching dilemma. They could take a choice between discarding the time-tested Copernican theory that the earth is in motion or the not so venerable, but just as scientifically respectable, ether theory which so elegantly took care of the be-

havior of light, electricity, and magnetism. It was at this point that the young German mathematical physicist, Albert Einstein, came upon the scene. He dared question that which had been accepted as obvious for two hundred years by not only rejecting the ether theory and with it the concept of space as a fixed framework of reference, but also by discarding the concept of absolute time.

The Michelson-Morley experiment had established one fact beyond any question, the velocity of light is unaffected by the motion of the earth. Einstein seized upon this as a suggestion of the existence of a universal law which would incorporate the mechanical laws of all uniformly moving systems, and also the laws governing light and other electromagnetic phenomena. Thus his fundamental postulate became: *All the phenomena of nature, all the laws of nature, are the same for all systems that move uniformly relative to one another.* Thus space and time became mere forms of intuition as Einstein underscored the words uttered by Berkeley two centuries earlier, "All the choir of heaven and furniture of the earth, in a word all those bodies which compose the mighty frame of the world, have not any substance without the mind."

Concepts such as "now," "how far," "here," and "simultaneity" call for new interpretations. For example, what time is "now"? Suppose someone in Nashville, Tennessee, is talking by long-distance telephone to a person in London, England. If it is 6:00 P.M. Central Standard Time in Nashville, it is midnight, Greenwich Meridian Time in London. What time is "now" depends upon which person speaks of "now," that is, it depends on the *frame of reference* of the person speaking of "now." For another illustration, suppose there are two men, one A, sitting beside a railroad track, and another, B, riding atop a passing boxcar. As the car passes A, B starts to walk from one end of the car to the other. How far is it from the point where B started walking to the point where he stopped? The answer will depend on whether it is given by A or by B. If given by A, the distance might be one hundred yards, a quarter of a mile, or some other distance, depending on the length of the box car *and* the speed with which the car is moving. If given by B, the distance would be simply the length of the box car. In other words "how far" depends on the frame of reference of the person answering the question.

All measurements of time and distance became variable quantities. The effective tool for Einstein's deductions became a system of equations known as the Lorentz transformations, which preserved the velocity of light as a universal constant but modified all measurements of time and distance according to the velocity of each system of reference. Another postulate thus became that *the laws of nature preserve their uniformity in all systems when related by the Lorentz transformations.* Still a third fundamental law of nature which the Theory of Relativity reveals is: *The velocity of light is the top limiting velocity in the universe.*

These laws may seem strange in the context of classical physics. They do not, however, contradict the old laws. The world has simply become a four-dimensional, space-time continuum in the context of Riemannian non-Euclidean geometry. The new frame of reference is one in which the old concepts of space and time are the limiting cases which apply solely to the familiar experiences of man. The new theory provides the physicist and astronomer with more accurate computations of planetary motion than those provided by the Newtonian theory.

Probably the most often verified and the most fruitful to experimental physicists of all the principles of Relativity is the relativity of mass. As an illustration, a ball held by a person has a definite mass. When that ball is thrown, the mass of the ball increases with its speed. Among other values, this principle is of extreme importance in the study of the behavior of elec-

trons in many types of radio tubes and atom-smashers. Furthermore, from it Einstein derived what has been called the "most important and certainly the most famous equation in history": $E=mc^2$.

This equation is the symbolic way of saying that a given amount of energy is physically equal to a definite amount of mass. In addition to playing an important role in the development of the atomic bomb, this equation provides the answer to many of the long-standing mysteries of physics. At the halfway mark of the twentieth century, we stand at the midpoint between microcosm and macrocosm in man's brilliant endeavor to discover the full significance of the Pythagorean dictum: Number rules the universe.

Yet the fundamental mystery remains. The whole march of science toward the unification of concepts—the reduction of all matter to elements and then to a few types of particles, the reduction of "forces" to the single concept of "energy," and then the reduction of matter *and* energy to a single basic quantity—leads still to the unknown. The many questions merge into one, to which there may never be an answer: what is the essence of this mass-energy substance, what is the underlying stratum of physical reality which science seeks to explore?

Each new venture into the stream of scientific thought can expose new worlds to conquer, present new challenges to meet. What prophetic genius would hazard a guess as to the effect of modern curious inquiry on the culture of future generations?

Many minutes of your time and quite a few hours of my time have been spent in arriving at this point. What of it now that we are here? Among modern-day schoolmasters there are likely to be some who, with Holofernes of old, would say, "He draweth out the thread of his verbosity finer than the staple of his argument."

Gossamer thread and coarse-grained staple intertwine to strengthen the bond between the curiosity of forebears and the culture of progeny. As our forefathers struggled to weave the fabric of "right truth" out of the confusion of "wrong truths," they found renewed inspiration in the stray suggestion, the wandering word, or the errant echo as they delved deep in their imagination, each on his own personal premises. Out of the tough texture of persistent thought and the silken thread of unheralded opportunity they spun the indestructible tie that binds the curiosity of the past with the culture of the present.

Let us not forget our responsibility to our children and our children's children. Doubt is creative, curiosity is original, and truth is but "a brief holiday between two long and dreary seasons, during the first of which it was condemned as sophistry and during the second is ignored as commonplace." So let us not

. . . despise the little things
Which happen daily round us,
For some of them may chance take wings
To startle and astound us.

We must not, in self-complacency, everlastingly look back at the fast fading lines of today's sunset. Let us cast our thoughts to the morrow as, with Rudyard Kipling, we each and everyone,

. . . keep six honest serving men.
[They taught him all he knew.]
Their names are What and Why and When,
And How and Where and Who.

The pupils in Tommy's class made a list of all the words that apply to size—large, small, tiny, big, etc. Suddenly Tommy raised his hand and said, "Oh, we forgot the most important word—'King-size.'"—*Taken from* The Instructor.

National High School and Junior College Mathematics Club (M A Th)

A National High School Mathematics and Junior College Club, (M A Th), has been formed. This is the first time that such an organization has been established at the national level and it is anticipated that existing and future high school mathematics clubs, if properly qualified, will wish to join the new organization.

I. Purpose of the organization

This club is being formed to engender keener interest in mathematics, to develop sound scholarship in the subject and promote enjoyment of mathematics among high school and junior college students.

National policy is determined by a seven-person governing council consisting of three national officers (president, vice-president, and secretary-treasurer) and four governors elected for three-year terms. Officers and governors serve without remuneration. Two council members are nominated by The National Council of Teachers of Mathematics and two by the Mathematical Association of America. The remaining nominees are submitted by the nominating committee of the Club. Terms of office are three years.

The governing council will provide the following services:

1. Individual Membership Certificates and School Charters will be issued by the national Secretary-Treasurer.
2. Chapters will be given suggestions at regular intervals of topics suitable for discussion at chapter meetings, and of books, periodicals, films, tape recordings, mathematical devices and similar items of interest to the membership.
3. Chapters may obtain help in securing desirable outside speakers.
4. Local newspapers will be notified that a charter has been granted and of names of students elected as full members.
5. Insignia pins and buttons will be available from the L. G. Balfour Co.

II. School qualifications for chapters

Any high school, two-year junior college, or other academic institution giving training equivalent to one of these, may petition to have a chapter providing it meets the following minimum requirements:

1. At least three semesters of algebra, and two of geometry or their equivalent, and one semester of more advanced mathematics (or, in the case of junior colleges, courses for which these are prerequisites) must be offered. These requirements can *not* be fulfilled by courses in general mathematics, business mathematics, shop mathematics, or arithmetic.
2. During the two semesters preceding that in which a petition is submitted, the school must have employed at least one teacher whose primary teaching field is mathematics and *who has completed an undergraduate mathematics major or its equivalent* at an accredited college or university.
3. The Principal, or other chief administrative officer of the institution, must approve the petition.
4. An initial charter fee of $2.00 along with the regular initiation fee of 75 cents for each member, must accompany the petition for a charter. (These fees will be returned if the governing council votes that the instituion is ineligible for membership.)
5. A favorable vote from two-thirds (5) of the governing council shall be required to elect a chapter to membership. Each petitioning institution will be notified as soon as possible whether or not a charter has been granted.
6. The petition should be submitted (along with 6 carbon copies) on the form provided by the national office. The carbon copies may be on plain paper.

III. Qualifications for individual membership

The following minimum requirements for full and associate membership shall be common to all chapters. Each chapter shall have a faculty-student committee which will recommend possible members for the chapter's consideration. No student shall be recommended for consideration who does not meet the minimum qualifications. Additional requirements may be imposed by individual chapters.

(a) *Full Membership:* Senior High School students who have completed two semesters of algebra and two of geometry, or their equivalents, and in addition have completed or are enrolled in intermediate algebra (third semester algebra) are eligible for full membership providing their mathematical work was done with high distinction and their general high school work with distinction. (On the ABCDF grading scale, this shall mean a B-plus and a B average, respectively.)

Senior high school graduates whose mathematical and general scholarship is at least equal to that required for undergraduates are eligible for full membership.

(b) *Associate Membership:* Persons who have completed two semesters of algebra or their equivalent with high distinction and who are enrolled in or have completed a semester of geometry are eligible for associate membership. *Associate members do not pay the 75 cents initiation fee* and are not registered with national headquarters. They are not entitled to vote on national policy. They are entitled to attend and be heard at meetings, and presumably are likely candidates for full membership.

IV. National finances

There shall be *no* national annual dues. An initiation fee of 75 cents per full member shall be paid to the national secretary-treasurer for each person initiated into full membership, whereupon the secretary-treasurer shall issue a membership certificate to that member and place his name on the official roll. A $2.00 charter fee shall be charged each new chapter at the time the official charter is first issued to the school.

The secretary-treasurer is charged with the handling of national funds, with an annual accounting to be made to the governing council.

V. Local organization, officers, and finances

Each chapter is free to set up its own organization, officers, and finances with the following restrictions:

1. Each chapter must have a semi-permanent faculty sponsor and the national office (secretary-treasurer) must be kept informed of the current faculty sponsor's name and address.

2. The minimum membership requirements set forth by the national office must be met by all initiates.

3. A complete and accurate list of all full member initiates, accompanied by their 75 cents initiation fees, must be received by the office of the secretary-treasurer *before* the initiation takes place.

4. Each chapter must hold regular meetings at periodical intervals and not merely consider itself an honor society for high grades. At the minimum, there should be one meeting every month.

5. Local chapters are encouraged to participate actively in the life of the school, providing stimulation for the better students and, if the members so desire, help sessions for those who need them.

6. Chapters preferring the name Mu Alpha Theta (M A Th) to N. H. S. & J. C. M. C. may use it.

VI. Governing council

President: Henry L. Alder, Department of Mathematics, University of California, Davis, California.

Vice-President: Edward L. Walters, Head, Mathematics Department, William Penn Senior High, York, Pennsylvania.

Secretary-Treasurer: Josephine P. Andree, Box 1127, The University of Oklahoma, Norman, Oklahoma.

Governors-General: Nellie M. Kitchens, Head, Department of Mathematics, Hickman High School, Columbia, Missouri; John R. Mayor, Director, Science Teaching Improvement Program, A.A.A.S., 1515 Massachusetts Avenue, N.W., Washington 5, D. C.; Virginia Lee Pratt, Mathematics Department, Central High School, Omaha, Nebraska.

(To secure a petition for Charter form, or other information, write directly to Josephine P. Andree, National Secretary-Treasurer, Mu Alpha Theta, Box 1127, The University of Oklahoma, Norman, Oklahoma.)

"The practical value of every social invention or material discovery depends upon its being adequately interpreted to the masses. The future of scientific progress depends as much on the interpretative mind as it does upon the creative mind. . . . The interpreter stands between the layman, whose knowledge of all things is indefinite—and the scientist, whose knowledge of one thing is authoritative. . . . The scientist advances knowledge. . . . The interpreter advances progress. . . . History affords abundant evidence that civilization has advanced in direct ratio to the efficiency with which the thought of the thinkers has been translated into the language of the masses."—*Glenn Frank, late President of the University of Wisconsin.*

• HISTORICALLY SPEAKING,—

Edited by Phillip S. Jones, University of Michigan, Ann Arbor, Michigan

American doctoral dissertations on mathematics and astronomy written by women in the nineteenth century

by Walter Crosby Eells, U.S. Office of Education, Washington, D.C.

The first doctorates of philosophy earned in an American university were conferred on three men at Yale University in 1861. The first doctorate earned by a woman was conferred at Boston University in 1877. Since 1861, as shown by tables prepared by the writer for the current edition of *American Universities and Colleges* (Washington: American Council on Education, 1956, seventh edition, pp. 65–80), more than 130,000 doctorates have been conferred in the United States, over 15,000 of them on women.

Records regarding these women, particularly prior to 1901, are very incomplete and inaccurate in the published reports of the U. S. Commissioner of Education. Considerable information concerning 228 of those who earned their doctorates during the nineteenth century is given in an article by the writer in the Winter 1956 issue of the *Bulletin* of the American Association of University Professors. This information includes names, institutions, dates, and major fields of study for these 228 women who thus were pioneers in the field of advanced scholarship in the United States.

Through a search of the early catalogs and other institutional documents in the library of the United States Office of Education, and through correspondence with the librarians, archivists, and alumni secretaries of the institutions concerned, the writer has also been able to secure the dissertation topics for more than nine-tenths of these women. There was not space however, to list these dissertations and other information in the *Bulletin* article referred to above. Information regarding publication of dissertations and in some cases birth and death dates of their authors has been found in the Library of Congress.

Of the entire group of 228 known nineteenth-century women doctors, at least 11 wrote dissertations in the field of mathematics or astronomy. Four of this group of dissertations were written at Yale University; two each at Bryn Mawr College, Columbia University, and Cornell University; and one at Carleton College. The earliest was accepted at Columbia University in 1886. Publication data have been found for more than half of them.

The names of these women (and subsequent married name, if any), with institution and year of their bachelor's degree, institution and year of their doctor's degree, and birth and death dates, as far as these data have been found, are given below. These facts are followed by the titles of their doctoral dissertations and all available information concerning

their publication.

This historical information concerning early women workers and their original contributions in the fields of mathematics and astronomy may be of some interest to those engaged today in the teaching of mathematics and in the direction of mathematical research in American universities and colleges.

Barnum, Charlotte Cynthia. (A.B., Vassar, '81), Ph.D., Yale University, 1895. Dissertation: "Functions Having Lines or Surfaces of Discontinuity."

Dickerman, Elizabeth Street, 1872– . (A.B., Smith, '94), Ph.D., Yale University, 1896. Dissertation: "Curves of the First and Second Degree in *X*, *Y*, *Z*, where *XYZ* are Conics Having Two Points in Common."

Edgerton, Winifred (Mrs. Frederick James Hamilton Merrill). (A.B., Wellesley, '83), Ph.D., Columbia University, 1886. Dissertation: "Multiple Integrals."

Furness, Caroline Ellen, 1869– . (A.B., Vassar, '91), Ph.D., Columbia University, 1900. Dissertation: "Catalogue of Stars within One Degree of the North Pole and Optical Distortion of the Helsingfors Astro-Photographic Telescope Deduced from Photographic Measures." Publication: *Publications of the Vassar College Observatory, No. 1*, Poughkeepsie, New York: 1900. iv, 74 pp.

Gentry, Ruth, 1862– . (Ph.B., Michigan, '90), Ph.D., Bryn Mawr College, 1896. Dissertation: "On the Forms of Plane Quartic Curves." Publication: New York: Press of Robert Drummond, 1896. 73 pp. and diagrams on 12 plates. Includes biography of author.

Lewis, Anna Delia. (A.B., Carleton College, '89), Ph.D., Carleton College, 1896. Dissertation: "Variable Stars." (*Note:* Miss Lewis is still living, or was on July 16, 1956. On that date she wrote from St. Paul, Minnesota, in part as follows: "Your letter . . . was received this morning. It carries me back over sixty years and I reply with sincere appreciation. You ask for the subject of my doctor's thesis: it was 'Variable Stars.' As you doubtless know, it was a timely subject in the 90's, when that study was in its infancy. I'm quite sure that there is a copy in Goodsell Observatory of Carleton College. . . . When I was in College (1885–1889), Dr. W. W. Payne was head of Goodsell Observatory for which he had great ambition, hoping to make it a School of Astronomy with a graduate course leading to the doctor's degree. By 1908 six people, three men and three women, had finished the course. Then a new president came to the College, who did not know (or care) about Professor Payne's plans, and the course was discontinued, and Dr. Payne retired. Goodsell is still one of the finest observatories in the country (for a college). Of the women who took all or a part of the course, one went to Smith College, Miss Mary Byrd; one, Miss Flora Harpham, to Columbia; and one, Miss Anne Young, is now Professor Emeritus of Astronomy at Mount Holyoke, while I was there for three years—she for forty. She and I are the only ones now living." Miss Lewis taught at different periods astronomy, mathematics, and physics at Carleton College, Albert Lea College, Mount Holyoke College, and Lake Erie College, retiring from the last named in 1936.)

MacKinnon, Anne Louise (Mrs. Edward Fitch). Ph.D., Cornell University, 1894. Dissertation: "Concomitant Binary Forms in Terms of the Roots." Publication: Charlottesville, Virginia: 1898. Reprinted from *Annals of Mathematics*, IX: 95–157, May–June, 1895; and XII: 95–109, 1898. "The part reprinted from vol. 9 was presented to the faculty of Cornell University in partial fulfilment of the requirements for the degree of

doctor of philosophy; that from vol. 12 is supplementary and contains tables."

MADDISON, ISABEL. (B.Sc., University of London, '93), Ph.D., Bryn Mawr College, 1896. Dissertation: "On Singular Solutions of Differential Equations of the First Order in Two Variables, and the Geometrical Properties of Certain Invariants and Covariants of Their Complete Primitives." Publication: *Quarterly Journal of Mathematics*, XXVIII: 311–374, 1896.

METCALF, IDA MARTHA. (M.S., Cornell University, '89), Ph.D., Cornell University, 1893. Dissertation: "Geometric Duality in Space." Publication: Ithaca, New York: E. D. Norton, Printer, 1893. 24 pp.

PALMER, MARGARETTA, 1862–1924. (A.B., Vassar, '87), Ph.D., Yale University, 1894. Dissertation: "Determination of the Orbit of the Comet 1847 VI." Publication: New Haven, Connecticut: *The Observatory*, 1893. (Transactions of the Astronomical Observatory of Yale University, I, pt. IV, 183–207.)

PIERCE, LEONA MAY. (A.B., Smith,'86), Ph.D., Yale University, 1899. Dissertation: "On Chain-Differentiants of a Ternary Quantic."

Not included in the above list, but worthy of note to complete the record of doctoral work by women in the nineteenth century, is the name of CHRISTIAN (LADD) FRANKLIN (Mrs. Fabian Franklin), of Johns Hopkins University, who, according to a letter from the alumni office of that institution, September 10, 1956, "was the first woman to complete work toward a [doctor's] degree (1878–1882) but the degree was not conferred upon her until 1926." Mrs. Franklin's dissertation, "On the Algebra of Logic," was completed in 1882 and appeared in the volume *Studies in Logic*, "By members of the Johns Hopkins University," edited by Charles S. S. Pierce, pp. 17–71. (Boston: Little, Brown & Co., 1883. 203 pp.)

Women in American mathematics—20th century

by Phillip S. Jones

The chief purposes of this note are to cite several outstanding women mathematicians of the current century and to suggest that a new study of the role of women in mathematics today would be very worthwhile. I hope some reader will undertake it.

The pamphlet *The Outlook for Women in Mathematics and Statistics* should be known to every teacher and counsellor, but it is a little incomplete and out-of-date, so it is good that it has now been succeeded by a new edition.[1]

Many parts of what is said about opportunities for mathematicians in the publications of the Mathematical Association of America and the National Council of Teachers of Mathematics[2] are today as

[1] *The Outlook for Women in Mathematics and Statistics*, U. S. Department of Labor, Women's Bureau No. 233–4, 1948. *Employment Opportunities for Women Mathematicians and Statisticians*, 1956 [published 1957]. 25¢. Catalogue No. L13.3:262.

[2] *Guidance Pamphlet in Mathematics for High School Students*, National Council of Teachers of Mathematics, 1953. *Professional Opportunities in Mathematics*, Mathematical Association of America, 1954.

true for women as for men. However, none of these sources tell of the increased numbers of women doing graduate work for Ph.D's in mathematics and of the opportunities which they find open to them.

Some data on these women could be obtained by an enterprising searcher by scanning the annually published list of persons winning doctorates in mathematics and following up the careers of the women, as well as by scanning the membership lists of the mathematical societies. The recent publication, *Employment After College: Report on Women Graduates, Class of 1955*, deals only with persons receiving the bachelor's degree, but its introduction states:

"Six months after graduation four-fifths of the graduates had jobs, most of them in the fields for which they had been trained. The majority had prepared themselves to teach, and 6 out of 10 of the employed graduates held teaching positions. Other graduates with jobs directly related to their field of study included those who had majored in nursing, biological sciences, chemistry, home economics, and *mathematics*" (italics mine).[3]

In spite of this statement, a later table shows that 10 per cent of the women with undergraduate majors in mathematics were seeking work[4]—more than for any other one field! Perhaps this is due to the need for additional study to qualify for good jobs as mathematicians. However, this report noted that the highest percentages of 1955 women graduates continuing on in graduate study were found among those who had majored in natural sciences (35 per cent of the physical science majors), and that less than 5 per cent of those who had majored in education, nursing, mathematics, physical education, business and commerce were graduate students!![5] Perhaps, on the other hand, this is because there is not enough information available to undergraduate women about opportunities in mathematics and how to seek them, since this report also points out, "Relatively few women secured training which could be utilized in shortage occupations other than teaching. For example, about 5 per cent had majored in nursing and other health fields; 3 per cent in biological sciences; and 2 per cent each in physical sciences and in *mathematics*—all shortage areas needing more trained people."[6]

Almost three fourths of the women had taken some work in education, and most of these held teaching jobs, but only 7 per cent of those holding secondary school certificates were certified to teach mathematics. (The only area with fewer new women teachers was modern languages!)[7] However, 9 per cent of those teaching secondary school *were* teaching mathematics! Of employed women with undergraduate majors in mathematics, 53 per cent were teaching and 32 per cent were working as mathematicians and statisticians.[8]

It is some consolation to note, however, that "the best paying jobs were held by chemists (averaging $3,900) and mathematicians and statisticians ($3,850).[9] This no doubt accounts for the fact that the average annual salary of graduates with mathematics majors ($3,402) was exceeded only by those of women with majors in health fields and the physical sciences.[10]

To return to women with advanced degrees in mathematics who have worked in America, Emmy Noether (1882–1935) is with little doubt the most outstanding. It is, of course, a little unfair to claim her as an American mathematician, since it was only the last year-and-a-half of her life which she spent at Bryn Mawr after leaving, with many other fine mathematicians,

[3] *Employment After College: Report on Women Graduates, Class of 1955*, U. S. Department of Labor, Women's Bureau, 1956.

[4] *Ibid.*, Table 7.

[5] *Ibid.*, pp. 8–9.

[6] *Ibid.*, p. 7.

[7] *Ibid.*, p. 13.

[8] *Ibid.*, Table 8.

[9] *Ibid.*, p. 14.

[10] *Ibid.*, Table 11.

that center of mathematical learning and research, Goettingen, under the destructive pressure of Hitler's Nazi party. She had gone to Goettingen from Erlangen in 1916. There she collaborated with such outstanding persons as Klein and Hilbert, who argued for her "habilitation" there by declaring in a faculty meeting, "I do not see that the sex of the candidate is an argument against her admission as Privatdozent. After all we are a university and not a bathing establishment."[11] In spite of this she was not given a title or recompense for her work until after the war in 1919. Listed among those who listened to her lectures in modern algebra, studied under her, or worked with her are: Hermann Weyl, John von Neumann, Emil Artin, Richard Brauer, all persons of the highest rank who later emigrated to this country. Other mathematical giants associated with her were Hasse, B. L. van der Waerden, and Alexandroff.

To try to summarize her mathematical work would not be appropriate to our purposes nor possible within the space available. However, I would recommend a study of her life and the lives and works of other women mathematicians as interesting material for club topics and class reports.[12]

Probably the outstanding woman mathematician of today is Mina S. Rees, Dean of Hunter College since 1953. After receiving an A.B. from Hunter, an A.M. from Columbia University, and a Ph.D. from the University of Chicago, she continued her teaching and research in abstract algebras and division algebras at Hunter except for outstanding war-time government service. This service, for which she was awarded the King's Medal in England in 1948 and a Presidential Certificate of Merit, had many aspects, typical of which were her terms (1946–1949) as head of the mathematics branch of the Office of Naval Research, and as Director of its Mathematical Sciences Division (1949–1953).[13]

Since Bryn Mawr was mentioned several times in the preceding article by Walter Crosby Eells and again in our account of Emmy Noether, we will conclude by again suggesting that some one of our readers should continue Mr. Eells' work by studying the women mathematicians of the twentieth century, and that in so doing this women's college will be noted not merely as Emmy Noether's haven, but also as having continued a tradition of vigorous interest in advanced mathematics with the work of Professor-Emeritus Anna Pell-Wheeler and Professor Marguerite Lehr.

[11] This quotation and most of the data about Emmy Noether are drawn from a Memorial address by Hermann Weyl published in *Scripta Mathematica*, III (July 1935), 201–20.

[12] See also: E. T. Bell, *Men of Mathematics*, Chap. 22, "Master and Pupil" (Weierstrass and Sonja Kowalewski), (New York: Simon and Schuster, 1937). Sister Mary Thomas a Kempis, "The Walking Polyglot," *Scripta Mathematica*, VI (Dec. 1939), p. 211 ff. "Caroline Herschel," *Scripta Mathematica*, XXI (Dec. 1955), p. 237 ff. A. W. Richeson, "Mary Somerville," *Scripta Mathematica*, VIII (Mar. 1941), p. 5 ff. "Hypatia of Alexandria," *National Mathematics Magazine*, XV (Nov. 1940), p. 74 ff. Carolyn Eisele, "Lao Genevra Semons," *Scripta Mathematica*, XVI (Mar.–June 1950), p. 22 ff. Julian L. Coolidge, "Six Female Mathematicians," *Scripta Mathematica*, XVII (Sept.–Dec. 1951), p. 20. R. C. Archibald, "Topics for Club Programs—Women as Mathematicians," *American Mathematical Monthly*, XXV (Mar. 1918), p. 136 ff.

[13] *American Men of Science 1955*, vol. I. Physical Sciences.

I said that a mathematician was a maker of patterns of ideas, and that beauty and seriousness were the criteria by which his patterns should be judged.—*G. H. Hardy.*

Edited by Lucien B. Kinney, Stanford University, and
Dan T. Dawson, Stanford University, Stanford, California

An observation on checking division

by R. F. Crawford, Laboratory School, Louisiana State University, Baton Rouge, Louisiana

Most teachers of arithmetic at the junior-high level emphasize the importance of checking computations. In division the most common recommendation is that the student should multiply the divisor by the quotient, then add the remainder. If this work is correct, the resulting number should equal the dividend. A second, quite laborious, method is to replace the divisor by the quotient and repeat the entire division process, in the hope that the number which formerly served as divisor will now appear as quotient. Still another method is "casting out 9's."

Layton Cox, a student in ninth-grade general mathematics, recently hit upon a scheme of checking which showed a great deal of ingenuity and which, incidentally, has been adopted by his classmates as a time saver.

To illustrate, consider the case when 8624 is divided by 35. This yields a quotient of 246, with a remainder of 14. The standard checking procedure would be: 246 (the quotient) ×35 (the divisor) yields 8610. Adding the remainder, 14, gives 8624, the original dividend. The same result, of course, would be obtained by multiplying the divisor by the quotient and adding to the product.

Layton discovered that the partial products in the latter check were identical with the partial products of the division exercise but in reverse order.

$$
\begin{array}{rcr}
246 & & \\
35/8624 & & 35 \\
70 & & 246 \\
 & & \overline{210} \\
162 & & 140 \\
140 & & 70 \\
\overline{224} & & \overline{8610} \\
210 & & +14 \\
\overline{14 = R} & & \overline{8624}
\end{array}
$$

This led him to realize that the necessary multiplication had already been performed. Hence, a simplified check can be made by adding the partial products from the long division if sufficient care is used in keeping them in proper columns.

Layton solved the question of proper columns, also. His method is to write the last partial product first, then move one space to the left for the right hand margin of each succeeding partial product. Layton's check becomes:

$$
\begin{array}{r}
210 \\
140 \\
70 \\
\hline
8610 \\
+14 \\
\hline
8624
\end{array}
$$

This is a useful technique for stimulat-

ing interest in what can become rather dull drill work. With very little teacher guidance the pupils can be led to discover this method of checking for themselves. A weakness of this method is that a multiplication error in determining one of the partial products (assuming this is the only error) is not detected in the checking process. An interesting lesson could be developed about this weakness—is it true? and why?

Editor's Note: The type of analysis made by this ninth-grade student may lead to many discoveries of relationships. They are particularly important if the "why" of a process is to be understood by the learner. "Checking" or "making sure" when viewed in this light can be a much more functional process than the little-understood *abracadabra* of "multiply the quotient by the divisor and add the remainder."

Trigonometry in grade eight

by Ramon Steinen, Horace Mann School, New York, New York

Few would deny that an important concern of the mathematics teacher in grades seven and eight is the further development of computational techniques introduced in the first six grades. That it is the most important objective is questionable. Rather than just "tying up loose ends" for two years, teachers should be doing a more efficient job of sowing new ideas and helping students build up a large, accurate vocabulary of mathematical terms.

Most textbooks written for use in grade eight contain material on congruence, similarity, and scale drawing. In some respects a unit on trigonometry can be as meaningful to the eighth grade student as any of these and more appropriate. (Trigonometry, for example, involves much more meaningful arithmetic.) The following unit was received with great enthusiasm by my eighth grade class.

Every student was armed with protractor, ruler, sharp pencil and scrap paper. Each student was instructed to draw a line segment on his paper; those in row one made a two-inch segment, those in row two a three-inch segment, and those in row three a four-inch segment. With this segment as base each student made a right triangle with a 40° acute angle. The legs of the right triangle were measured and the ratio of the leg opposite the 40° angle to the original line segment was found. The decimal equivalent of this fraction was found correct to the nearest hundredth. Each student's result was written on the board. Since all the students were using rulers graduated in the same units, the decimal results tended strongly to agree within each row but differ slightly from row to row. Each student was then instructed to make another right triangle with a 40° angle and any length base he pleased. The same computation was done as before. When the results were tabulated on the board, the class was able to agree that the decimal values all seemed to be about the same, even though the triangles were of varying sizes. Over two-thirds of the class obtained a value of .83 or .84. They were willing to accept my suggestion that .84 be considered the more accurate of the two.

No name had been given to this ratio as yet—I felt it better to go into some of its uses first. Naturally, I had several students ask what this was for, how it could be used, why they were doing it, etc. Instead of an explanation I proposed a problem. "If you were standing one hundred feet from the side of a building and found that the angle you had to look up at

to see the top was 40°, how high would the building be?" Almost before I finished the question several cries of "84 feet" rang out. I might have been content to accept this, but one thoughtful fellow pointed out that it would be 84 feet more than the observer's height. We drew a sketch on the board to illustrate what he meant.

Most of the class saw readily where the 84 feet came from. We discussed it long enough to satisfy everyone as to its origin. What bothered several of them was how the angle in this problem could be measured. However, one student knew about the use of a transit and his contribution settled that problem.

A second problem was then proposed. "If this flag pole is forty feet tall and its shadow is fifty feet long, about how big is angle X?" "40°!" they cried. "Exactly?" I asked. "No." "Larger or smaller?" "Smaller." "Why?" "Four-fifths is less than .84."

"One more problem. You are standing on the edge of a pond. On the other side are two trees fifty feet apart—one of them is exactly opposite you. If you face one tree and then turn and face the other you must turn an angle of 50°. How far is it from you to the nearer tree?" This one gave them trouble until I asked them what they knew about the angles of a triangle.

Now it was time to name the creature. They agreed that it was a ratio. When informed it was called the tangent ratio, several of them looked pleased and one asked if that wasn't trigonometry. When I said it was, what I saw before me was one of the most pleased classes I have every seen. (I was to find out later that they were quick to inform their friends in other sections that they were studying trigonometry.)

The assignment for the first night was to construct a table of tangents for angles of 10°, 20°, 30°, · · · 80° correct to the nearest hundredth. These were compared the next day and we agreed on the correct set of values. The remainder of the period was spent solving right triangles. Given a leg and an acute angle of a right triangle they would find the complement of the given angle by subtraction, use the tangent ratio to find the other leg, and use the Pythagorean Theorem to find the hypotenuse. Those people who agree with me that computation, while not all important, is one of our primary concerns should consider how much arithmetic is involved in this one problem. First, a simple subtraction. Second, a division, with divisor and dividend usually of two or three digit numbers and the quotient required correct to the nearest hundredth. Third, squaring two numbers, adding, and finding the square root. Arithmetic skills are probably best developed in this manner. When these processes are used to solve problems which interest the students, the number of mistakes made is noticeably less than when a set of comparable drill examples is done.

We were able to do half a dozen different problems which I took from the trigonometry unit in a plane geometry text, keeping the same wording but changing the given angles to those in our tables. Then I assigned five problems and set them to work for the remaining period.

On the third day I was asked, "What do we do if we want to work with a triangle with a 37° angle—it isn't in our table?" I gave my answer in two parts. First, I showed them tables in other books: a four-place table with angles to the nearest degree and a four-place table with angles to the nearest 10 minutes. We used the former to solve a problem in which the angle was 37°. Then I asked them how many different-sized angles there were less than 90°. They all agreed there was no end to them—someone even said the number was infinite. They were able to see, then, that no matter how large a table they had it could not contain every possible angle. They had seen previously that one with angles correct to the nearest 10 minutes was fairly large.

I must confess that I had my doubts as to whether it would prove profitable or not,

but I decided to discuss interpolation with them.

"If a bus that always travelled the same speed stopped at Town A at noon and at Town B at 1 P.M., when would it pass a point halfway from A to B?" All agreed on 12:30 P.M. "If A and B are 20 miles apart, when does it pass a point 7 miles past A?" This took more thought, but one of the better students solved the problem and we discussed it until we all saw the answer.

"If the tangent of 30° is .58 and the tangent of 40° is .84, what is the tangent of 35°?" The students had little trouble solving this and offering .71 as the answer. It seemed logical to them that an angle halfway between 30° and 40° would have a tangent halfway between .58 and .84. Of course, their "logical" solution was not exactly correct. I showed them that according to a larger table the tangent of 35° should be .70 correct to hundredths. We did several examples using interpolation. One of these was the problem with the 37° angle we had done previously. We were able to compare the result using interpolation with that using the more complete table. We agreed that interpolation seemed reasonably accurate but also agreed to remember that it was not exact.

The better students were not content to stop here, but wanted to know how to find a more accurate value for an angle whose tangent fell between values in one table. Again, with only a few questions on my part, they were able to take the tangent of an angle and interpolate to find the angle "correct to the nearest degree." Due to the nature of the table their answers were not always correct to the nearest degree, but I felt justified in what we were doing since the process was correct.

That evening I gave an assignment that involved solution of right triangles by use of the tangent ratio and Pythagorean Theorem. Included in the assignment were problems that required interpolation to find the tangent given the angle and also interpolation to find the angle given the tangent. This was after four days on the unit.

The fifth day we reviewed all we had done, and on the sixth day they took the following test:

Angle	**Tangent**
10°	**.18**
20°	**.36**
30°	**.58**
40°	**.84**
50°	**1.19**
60°	**1.73**
70°	**2.75**
80°	**5.67**

In Figures 1 and 2 find the lengths of the sides represented by letters.

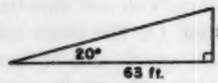


Figure 1

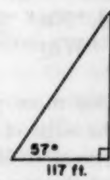


Figure 2

In Figures 3 and 4 find x *and* the number of degrees in each acute angle (to the nearest degree).

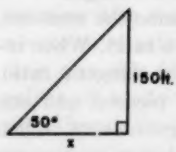


Figure 3

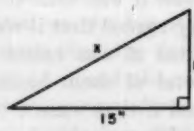
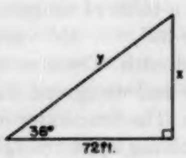


Figure 4

In Figure 5 find x and y to the nearest tenth of a foot.

Figure 5



Extra:

If you're feeling very bright, perhaps you can find the area of an equilateral triangle with 6-inch sides.

The only problem that presented real difficulty to most of them was the second part of #4. Only five out of eighteen were able to find the angle correct to the nearest degree. Of eighteen students that took the test eighteen solved the second, fifteen the third, thirteen found the hypotenuse in #4 and five the angles, eleven solved the fifth correctly and seven solved the extra credit problem. Only one student made an arithmetic error on this test!

We spent another week studying trigonometry. On the first day of this week I drew a large right triangle on the board and labeled the sides *a*, *b*, and *c*. We made six ratios with these letters and agreed that they were the only possibilities. I gave them the names of the ratios in terms of one of the acute angles. We discussed the fact that there were three pairs of reciprocal functions. I also pointed out what the "co" in cosine, contangent and cosecant meant.

Three days were spent working with the sine ratio. At the end of this time the following test was given:

Angle	Sine	Tangent
10°	.17	.18
20°	.34	.36
30°	.50	.58
40°	.64	.84
50°	.77	1.19
60°	.87	1.73
70°	.94	2.75
80°	.98	5.67

In Figures 6 to 10 find all angles to the nearest degree. Find all sides to the nearest tenth of a foot.

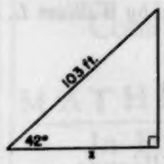


Figure 6

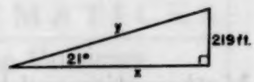


Figure 7

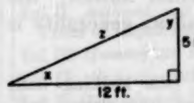


Figure 8

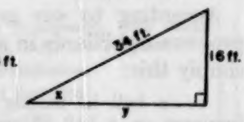


Figure 9

Figure 10

Again the results were good.

Two weeks were spent on this unit. During this time the students were introduced to the tangent and sine ratios and interpolation. Much meaningful arithmetic was done. We made use of the previously studied Pythagorean Theorem. The knowledge of the angle sum of a triangle and the meaning of complementary angles was put to use.

In the three years that I have taught eighth grade classes I have never had a unit of work received with more enthusiasm by a class. I am sure that this unit on trigonometry did more to create a real interest in mathematics than anything I had tried previously.

• MEMORABILIA MATHEMATICA

Edited by William L. Schaaf, Brooklyn College, Brooklyn, New York

Mathematics and billiards . . .

According to our genial author,[1] the problem of billiards in all its generality is simply this:

> . . . a ball in motion (the striking ball) impinges on a ball at rest (the object ball) or on a cushion, to find the subsequent path of each ball, in the first case, or of the striking ball in the second. . . . The motion of the striking ball (where no cushion is struck) may be of the following kinds: (1) *Perfect rolling;* (2) *Perfect sliding;* (3) *Imperfect rolling;* (4) *Pure side motion;* (5) *Curving motion.* These, with their combinations, exhaust all the possible modes of motion of the striking ball.

Each of these motions is then carefully defined and described in words; subsequently the equations for the several motions are developed. For example:

> *Perfect rolling* is the normal position of a billiard ball. In order to make a ball start with perfect rolling, it is only necessary to strike it horizontally at a point k^2/a above the centre, where a is the radius of the ball and mk^2 the moment of inertia about the centre. (Since both balls are of equal mass, the factor m is omitted, taking the mass of the ball as the unit of mass.) Since $k^2/a^2=\frac{2}{5}$ for a homogeneous sphere, this point will be about $\frac{7}{10}$ of the diameter of the ball above the table. This is just the height of the cushions, an arrangement without which cushion play would be impractical, as the table makers seem to have found out by rule of thumb.
>
> When a striking ball with perfect rolling impinges on the object ball at rest, the instantaneous result of the impact may be described by the equations of the velocities of the two balls at the instant when the impact has been completed, as follows:
>
> $$u_x=V\sin\theta-f\cdot\sin\theta\int Pdt;$$
>
> $$u_y=V\cos\theta-(1-\mu)\int Pdt;$$
>
> $$u_z=f\cdot\cos\theta\int Pdt;$$
>
> $$\bar{w}_x=\frac{V}{a}\cos\theta-\frac{a}{k^2}(f\cdot\cos\theta+\mu)\int Pdt;$$
>
> $$\bar{w}_y=\frac{V}{a}\sin\theta;$$
>
> $$\bar{w}_z=-f\frac{a}{k^2}\sin\theta\int Pdt.$$
>
> Here
>
> f = the coefficient of friction between balls;
> $1-e$ = the modulus of elasticity;
> 1 = the mass of the ball;
> P = the normal impulsive force operating so long as the balls are touching;
> θ = the angle between the direction of motion of the striking ball and the common normal;
> μ = some unknown small quantity depending upon the impulsive friction between the ball and the table;
> x-axis: the horizontal parallel common tangent;
> y-axis: the normal to both balls;
> z-axis: vertical upwards.
>
> The unknown quantity $\int Pdt$ may be eliminated by substitution, using the equality
>
> $$u_y=eV\cos\theta,$$
>
> which leads to the following equations:
>
> $$u_x=V\sin\theta\left\{1-\frac{f(1-e)}{1-\mu}\cdot\cos O\right\};$$
>
> $$u_y=V\cos\theta\left\{\frac{e}{1-\mu}\right\};$$
>
> $$u_z=V\cos^2\theta\left\{\frac{f(1-e)}{1-\mu}\right\};$$
>
> $$\bar{w}_x=\frac{V}{a}\cos\theta\left\{1-\frac{a}{k^2}\left(\frac{1-e}{1-\mu}\right)(f\cdot\cos\theta+\mu)\right\};$$
>
> $$\bar{w}_y=\frac{V}{a}\cdot\sin\theta;$$
>
> $$\bar{w}_z=-\frac{V}{a}\cdot\sin\theta\cos\theta\frac{f(1-e)}{1-\mu}\cdot\frac{a}{k}.$$

The bulk of the essay, including an appendix, is devoted to the derivation of

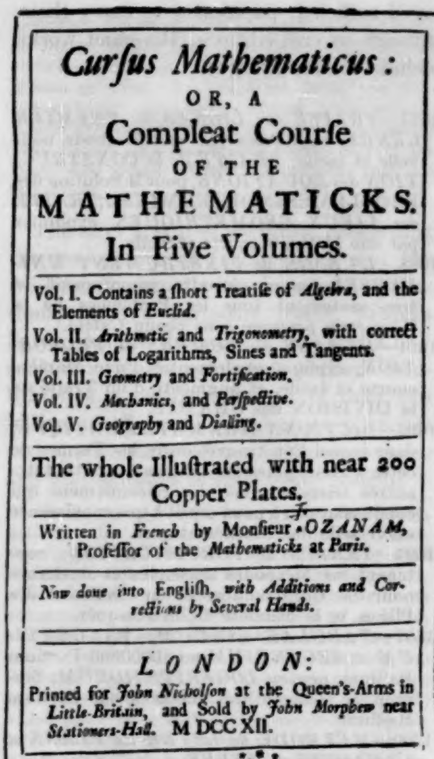
[1] G. W. Hemming, *Billiards Mathematically Treated.* (London: Macmillan & Co., 1899), 45.

similar equations of motion—not only for cases of sliding, side motion, curved motion, and combinations, but also for cases where the striking ball hits the cushion first. These equations become, as may be expected, rather elaborate. Of greater interest, perhaps, are the author's closing remarks:

> . . . there is no one of the conclusions to which science has led us . . . which has not been arrived at by rule of thumb by players who have not professed any scientific knowledge at all. It is interesting however to note that the conclusions from our scientific inquiry are absolutely in harmony with the established rules of practice. For comparatively untrained players, they have, moreover, some practical value. A rule of thumb is as good as a scientific law to a man who has played often and well enough to regard the rule of thumb as a necessary law of Nature. Amateurs of less experience than this may find it much easier to obey a law, the reason of which they have grasped, than to follow a rule merely because a highly-developed billiard-marker has told them to do so.

Jacques Ozanam on Mathematics . . .

The name of Ozanam is generally associated with his *Recreations in Mathematics and Natural Philosophy*. In a certain sense, he might well be considered a pioneer among writers on mathematical recreations and puzzles, since his original work was among the first to become widely known, and most writers since then have borrowed heavily from him. Yet he was by no means the first, for as early as 1612 the Frenchman Claude Gaspard Bachet de Méziriac published his *Problèmes plaisants et délectables qui se font par les nombres;* a second edition appeared in 1624. In this same year, there was published under the *nom de plume* of Van Etten a volume entitled *Récréations mathématiques*, the author of which was the Jesuit Jean Leurechon. General interest in such books apparently grew rapidly, for this was soon to be followed, in 1630, by Claude Mydorge's *Examen du livre des récréations mathématiques et de ses problèmes*. In 1636 Daniel Schwenter's *Deliciae physico-mathematicae oder Mathematische*


Curſus Mathematicus:

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Written in *French* by Monſieur OZANAM, Profeſſor of the *Mathematicks* at *Paris*.

Now done into Engliſh, *with Additions and Corrections by Several Hands*.

LONDON:

Printed for *John Nicholſon* at the Queen's-Arms in *Little-Britain*, and Sold by *John Morphew* near *Stationers-Hall*. M DCC XII.

und philosophische Erquickstunden appeared posthumously, and in the years 1641–42 the Italian Jesuit Mario Bettini issued the first two volumes of his *Apiaria universae philosophiae mathematicae in quibus paradoxa et nova pleraque machinamenta exhibentur*, to be followed in 1660 by a third volume under the title of *Recreationum mathematicarum Apiaria XII novissima*. On the heels of this came the *Arithmetische Lustgarten* of Johann Mohr, published in 1665. Thus by the time Ozanam's work on mathematical recreations first appeared in 1697, there had already accumulated a considerable precedent as well as a sizable demand for such works.

Jacques Ozanam in no sense of the word ever attained eminence as a professional mathematician. Yet he was the author of several mathematical books which en-

joyed sufficient popularity to carry them through several editions. His chief works include the following:

1687—*TRAITÉ des LIGNES du PREMIER GENRE*, expliquées par une Méthode nouvelle et facile; *TRAITÉ de la CONSTRUCTION des ÉQUATIONS*, pour la Solution des PROBLÈMES INDETERMINEZ; *TRAITÉ des LIEUX GÉOMETRIQUES*, expliquez par une Méthode courte et facile.

1688—*L'USAGE de l'INSTRUMENT UNIVERSEL*, pour résoudre promtement et très-exactement tous les Problèmes de la Géométrie pratique sans aucun Calcul.

1691—*L'USAGE du COMPAS de PROPORTION*, expliqué et demontré d'une Manière courte et facile, et augmenté d'un Traité de la DIVISION des CHAMPS.

1691—*DICTIONNAIRE MATHÉMATIQUE*, dans lequel l'on trouve, outre les Termes de cette science, plusieurs Termes des Arts et des autres sciences; avec des raisonnemens qui conduisent peu à peu l'esprit à une conaissance universelle des Mathématiques.

1694—*TRAITÉ de FORTIFICATION*, contenant les Méthodes anciennes et modernes pour la Construction et la Deffense des Places, et la Manière de les attaquer.

1697—*TABULAE SINUUM, TANGENTIUM* et *SECANTIUM* ad 10000000 Partium Radium, necnon *LOGARITHMORUM*, Sinuum ac Tangentium 10000000000 Partium Radium.

1700—*MÉTHODE de LEVER les PLANS et les CARTES de TERRE et de MER.*

1702—*NOUVEAUX ÉLÉMENS d'ALGÈBRE.*

1708—*RECREATIONS MATHEMATICAL AND PHYSICAL*; laying down and solving many Profitable and Delightful Problems, now done into English.

1712—*CURSUS MATHEMATICUS*: or a Compleat Course of the Mathematics [Parts I–V].

1757—*LA GÉOMÉTRIE PRATIQUE*, contenant la Trigonométrie théoretique & pratique, la Longimétrie, la Planimétrie, & la Stéreométrie. Avec un petit Traité de l'Arithmetique par Géométrie.

1769—*LA PERSPECTIVE THÉORIQUE* et PRATIQUE, ou l'on enseigne la Manière de mettre toutes Sortes d'Objets en Perspective, et d'en représenter les Ombres causées par le Soleil ou par une petite Lumière.

What is known of Ozanam's personal life is meager enough, although not without certain interest.[2] He was born in 1640 at Bouligneux, France. Although his family were of Jewish extraction, they had long been members of the Church of Rome, and Jacques, who was the younger of two sons, set forth to prepare himself for a career in the church. It soon became evident, however, that he had little inclination for theological studies. He was reputed to have been somewhat of a spendthrift, and addicted to rather gay living. Moreover, chemistry and mechanics began to attract his earnest attention, and so, a few years later, he abandoned all thought of the clergy and devoted himself to the study of science. His opportunities for such study were so limited, however, that he must be considered as a self-taught scientist. At first he did not regard his scientific activities as a means of livelihood; he soon moved to Lyons, where he taught mathematics gratuitously, considering it a degradation to accept money for such instruction. His gambling proclivities soon brought him to realize the sterner realities of life; and under the stress of financial embarrassment his attitude toward the teaching of science gradually changed. After some further years in Paris, spent largely in dissipation, he eventually settled down and married a young woman, also without means. It appears to have been a happy marriage, although all twelve children born of this union died while quite young, and he deeply lamented his wife's death in 1701. During these years he apparently derived a modest living in Paris by teaching mathematics, chiefly to foreigners. Losing most of his students with the advent of war, his circumstances again became rather precarious, until he was later admitted an *élève* of the Academy of Sciences. He died of apoplexy, in Paris in 1717, at the age of seventy-seven. According to Hutton, he was of a mild and cheerful temper, generous to the full extent of his means, and of an inventive genius; and his conduct after his marriage was irreproachable. He was devout, but adverse to disputations about points of faith. On this subject he

[2] Biographical data is based largely on Charles Hutton's English translation of Montucla's edition of Ozanam's *Recreations*.

used to say: "It is the business of the Sorbonne to discuss, of the Pope to decide, and of a *mathematician to go straight to heaven in a perpendicular line.*"

Such was the man who has left us the *Récréations*, the *Dictionnaire*, and the *Course of Mathematics* in five volumes. While this latter work seems not to be too well known, an anonymous English translation appeared in 1712; we reproduce herewith in facsimile the title page of the first volume. Each of the five volumes carries a lengthy preface. These naïve, intimate prefaces, which recapture with unusual poignancy the spirit of the times, would seem to merit preservation; accordingly we have reproduced selected excerpts, without troubling to identify them.

After so many Mathematical Works, that have been already Published, as well in the several Parts, as in a Body, usually call'd *a Course of Mathematics*, in imitation of those that had done the like in other Sciences; I shou'd never have entertain'd the least Thought of increasing the Number, and of composing a New *Cursus*, had not I found those hitherto done were but of little use. . . .

For every body courts the Mathematics, especially such of the Nobility and Great Men, as used to distinguish themselves by despising the Learning of the Schools, but are however charm'd with the Beauties of this Science.

The Necessity that Gentlemen are under, that would become considerable in the Art of War, or any great Employment, which cannot subsist without recourse to the Mathematics, makes them leave off several trifling Amusements, and apply themselves to these Sciences; and oftentimes the unexpected Pleasures they meet with, do so surprise and engage them, that they make it ever after as well the delightful as the serious part of their Studies.

One wou'd think the Weakness of Man was intended by Nature to incite him to the Study of *Mathematicks*. Other Animals She has sufficiently endow'd with Strength, Swiftness, and Offensive as well as Defensive Weapons; but has left Man altogether naked, and given him no other Portion, but Wit and Invention: With these he encreases his Strength, acquires Swiftness, defends himself against all Attacks and from all Injuries, is bold enough for any Undertaking, raises himself to the Heavens, Studies and Measures their Motions, and applies 'em to his own Use.

Sciences are now no longer a Mystery; for in this Age it is easy for any Man to become Learned; the most difficult Sciences are taught so many Ways, that it is impossible but we shou'd understand something of 'em, by some means or other: Geographical Maps are seen everywhere; any Body knows how to use Kalendars and Epacts, and even Handicrafts Men invent and make such things after their Way. And almost at every House one may see Sun-Dials.

The next group of excerpts, concerning the importance of fortifications and the relation of geometry to plans for fortifications, have a curiously ironic, modern flavor, despite the lapse of two and a half centuries, in an age when perhaps two-thirds of a national budget of over $70 billion is allocated for defense purposes and the shortage of manpower in science and mathematics is upon every tongue.

Since many Gentlemen study Geometry, with no other intent than for the better Understanding of the Art of War; I have, for that Reason, to this Geometry annexed Fortification, which has always been look'd upon as most honourable, as being the Business of a Hero, and is now in great Reputation. There are not many curious Things to be said concerning the Origin of Geometry, but there are abundance upon that of Fortification, and its Progress; which have been sufficiently mention'd by almost all those who treated of this Art: Therefore to avoid being tiresome by too long a Preface, I shall content myself with saying something of Geometry.

The knowledge of the Distances of Things has been always found so necessary, that Men could never have liv'd without considering them, and marking them in their Memories, or by some exterior Sign: So that tho' 'tis pretended the *Egyptians* were the first Inventers of Geometry, because they were the first that were under any Necessity of preserving the Limits and Extents of their Lands and Possessions, either by Figures traced and delineated, or by Accounts and Memorials contain'd in Writing, the Inundations of their Lands by the overflowing of the *Nile*, effacing all other Marks; yet as there was the same necessity before the inundated part of *Egypt* was inhabited, since there was no making any Division, without taking the Measures of the whole, and of each Part; nor any Building of Houses and Towns, without being appriz'd of their Figures, and regulating them by Measures; 'tis to be presum'd, that the Science of Measuring is as Old as the World. Man took from himself the Measures of all other Things; he did not borrow from others the Names of Fathoms, Cubits, Feet, Inches, he presently applied his Thumb, his Foot, his Hand, and his Arms to the Things he wou'd measure; and so he took from his own Body the Dimensions of all other Things, having no

occasion for Books or Masters, and without burthening the Memory with Names or Figures.

Fortification therefore is the chief Art wherein Geometry is used, and is at present in such a degree of Perfection, that there is no likelyhood it will receive any essential Addition; because it seems impossible to find any Thing that is not already invented for the attacking of Places. Earth is made use of for Trenches, Banks, and Retrenchments; Water for Sluces and Inundations; and Fire is applied various ways below, by the contrivances of Mines; above, by Granadoes, Bombs and Carkasses, and directly by Canons, and other pieces of Ordance. Methods of Defence have been invented against all these; and nothing is wanting but the perfecting them; and unless Mankind finds out some other Elements, there is not much probability that they can invent New Methods, either Offensive or Defensive.

I well enough know, 'tis not long since any very regular Fortifications have been made; and that, consequently, Geometry was not so very necessary to Engineers of old, as 'tis to those of the present time, who apply it with great Industry and Success, to the improvement of Military Architecture. This shews there is nothing to be done without Geometry, expecially, at present, when War is now grown a necessary Art, which every Body is obliged to learn, because of the unhappy Quarrels which are spread all over *Europe*.

Paperback mathematics . . .

Not long ago, a colleague of mine, a mathematics chairman in a large metropolitan high school, suggested the possibility that school libraries with limited appropriations, or schools with "petty-cash funds," might take advantage of inexpensive paper-bound editions of "popular" books on mathematics to augment their school library accessions, and also to supplement the resources of the mathematics club. . . . Are there many such books today? I was asked. Could I supply him with a list of titles? A little searching yielded the following list. It should be noted that paper-bound *textbooks*, as well as all sorts of *workbooks, drill books, review books*, etc. have been deliberately excluded. Prices in a few instances may not be entirely accurate, and in any event are always subject to change.

Although we have reason to believe that the list is fairly complete, a few titles undoubtedly have escaped our attention, and fail to appear for no other reason than that our search was not sufficiently diligent. Authors and publishers of such books will please forgive, but please not forget to tell us about, the omission so that we may add such titles.

Abbott, E. A., *Flatland*. Dover. $1.00.

Armitage, Angus. *The World of Copernicus* (Sun, Stand Thou Still). Mentor Books, #M65. 35¢.

Barnett, Lincoln. *The Universe and Dr. Einstein*. Mentor Books, #M71. 35¢.

Bishop, Calvin C. *The Slide Rule and How to Use It*. Barnes & Noble. $1.25.

Bonola, R. *Non-Euclidean Geometry*. Dover. $1.95.

Cantor, George. *Transfinite Numbers*. Dover. $1.25.

Dantzig, Tobias. *Number: The Language of Science*. Anchor Books, #A67. 95¢.

Descartes, René. *The Geometry*. Dover. $1.50.

Einstein, Albert, *et al. The Principle of Relativity*. Dover. $1.65.

Gaines, Helen F. *Cryptanalysis*. Dover. $1.95.

Gamow, George. *One Two Three . . . Infinity*. Mentor Books, #Ms97. 50¢.

Gardner, Martin. *Fads and Fallacies*. Dover. $1.35.

Gardner, Martin. *Mathematics, Magic and Mystery*. Dover. $1.00.

Hadamard, J. *The Psychology of Invention in the Mathematical Field*. Dover. $1.25.

Heath, R. V. *Math-e-Magic: Magic, Puzzles, and Games with Numbers*. Dover. $1.00.

Kamke, *Theory of Sets*. Dover. $1.35.

Kaufman, Gerald. *The Book of Modern Puzzles*. Dover. $1.00.

Klein, Felix. *Elementary Mathematics from an Advanced Standpoint: Algebra*. Dover. $1.75.

Klein, Felix. *Elementary Mathematics from an Advanced Standpoint: Geometry*. Dover. $1.75.

Klein, Felix. *Famous Problems of Elementary Geometry*. (Trans. by W. W. Beman and D. E. Smith.) Dover. $1.00.

Klein, Felix. *Lectures on the Icosahedron, etc.* Dover. $1.85.

Kojima, Takashi. *The Japanese Abacus: Its Use and Theory*. Chas E. Tuttle Co., Rutland, Vt. $1.25.

Kraitchik, Maurice. *Mathematical Recreations*. Dover. $1.65.

Leopold, Jules. *Check Your Wits*. Popular Library, Inc., #315. 25¢.

Lucey, R. M. *A Problem a Day*. Penguin Books, #866. 2 shillings.

Manning, H. P. *Geometry of Four Dimensions*. Dover. $1.95.

Merrill, Helen A. *Mathematical Excursions*. Dover. $1.00.

Meyer, Jerome. *Puzzle, Stunt, Quiz, Fun*. Dover. $1.00.

Mott-Smith, Geoffrey. *Mathematical Puzzles*. Dover. $1.00.

RANSOM, WILLIAM R. *Calculus Quickly*. The Author: 13 Barrows Road, Reading, Mass. 60 p. 1956.

READ, A. H. *A Signpost to Mathematics*. London: C. A. Watts; Thrift Books, #8. 1 shilling.

RUSSELL, BERTRAND. *An Essay on the Foundations of Geometry*. Dover. $1.50.

SARTON, GEORGE. *Study of the History of Mathematics*. Dover. $1.50.

SAWYER, W. W. *Mathematician's Delight*. Penguin Books, #A121. 50¢.

SAWYER, W. W. *Prelude to Mathematics*. Penguin Books, #A327. 65¢.

SCHAAF, W. L. *Mathematics for Everyday Use*. Barnes & Noble. $1.35.

SMITH, LAURENCE D. *Cryptography*. Dover. $1.00.

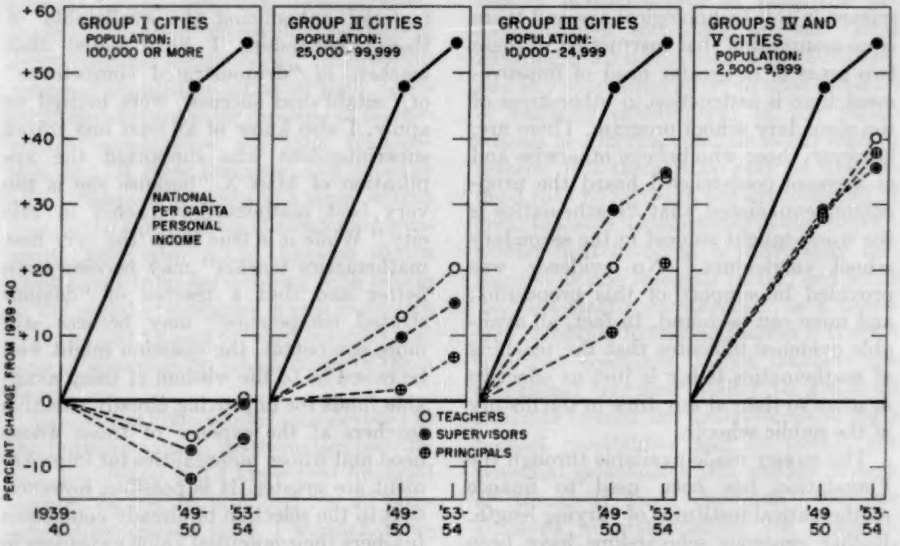
STRUIK, DIRK J. *A Concise History of Mathematics*. Dover. $1.60.

VINOGRADOV. *Elements of Number Theory*. Dover. $1.60.

WILLIAMS, W. T. and SAVAGE, G. H. *The Penguin Problems Book*. Penguin Books, #260.

YOUNG, J. W. A. *Monographs on Topics of Modern Mathematics*. Dover. $1.95.

Percent change from 1939–40 of (1) average salaries of instructional staff in public elementary and secondary schools in cities of different size and (2) national per capita personal income—*all in terms of dollars with 1953–54 buying power* (according to Consumer Price Index)—in 1949–50 and 1953–54



—*Taken from* School Life (*November, 1956*), *official journal of the Office of Education, Washington, D. C.*

• POINTS AND VIEWPOINTS

A column of unofficial comment

Better mathematics teaching—where?

by Harold Fawcett, Ohio State University, Columbus, Ohio

The National Science Foundation is providing large sums of money for the purpose of improving the teaching of mathematics and science. The availability of these funds for such a worthy purpose is a reflection of the fact that the Foundation recognizes the very great importance of science in this technological age and there is no assumption that instruction in these two areas is in greater need of improvement than is instruction in other areas of the secondary school program. There are, however, those who believe otherwise and at a recent conference I heard the proposition enunciated that "mathematics is the worst-taught subject in the secondary school curriculum." No evidence was provided in support of this proposition and none can be found. In fact, all available evidence indicates that the teaching of mathematics today is just as effective or more so than at any time in the history of the public schools.

The money made available through the Foundation has been used to finance mathematical institutes of varying length. Rather generous scholarships have been provided for those who qualify and during recent summers an increasing number of mathematics teachers have profited from this inviting opportunity. Who are these teachers? How are they selected? What criteria must be met to "qualify" for such a scholarship? These are not idle questions, for the extent to which such an improvement program really does improve mathematics instruction depends in considerable measure on how they are answered. I am, of course, not familiar with the process or the criteria used in screening the applicants for the several institutes already held, but in reading some of the published material announcing the availability of these scholarships I have noted that teachers of "demonstrated competence" or "established success" were invited to apply. I also know of at least one school superintendent who supported the application of Miss X "because she is the very best mathematics teacher in this city." While it is true that "the very best mathematics teacher" may become even better and that a teacher of "demonstrated competence" may become still more competent, the question might well be raised as to the wisdom of using available funds for improving already effective teachers at the expense of those whose need and whose potentialities for improvement are greater. It is possible, however, that in the selection of already competent teachers their potential value as leaders in their own schools may have been an influential factor. How helpful it would be, for example, if these teachers of "established success" could return to their respective local situations and there provide a quality of leadership such as to raise the level of all mathematics instruction in that area. While there is, as yet, no

indication that such in-service functions have been formally planned, I hope they may be anticipated, for such provisions could be effective in assisting other teachers of less experience and background.

It should be noted, however, that the present widespread criticism of mathematics instruction does not find its origin in the classrooms of teachers of "demonstrated competence." It stems from the classrooms of sincere and earnest teachers who, nevertheless, have never captured the spirit of mathematics, who really do not understand the nature of the subject for which they have teaching responsibility, and who fail to appreciate the significance of mathematics in modern culture. Any permanent improvement in mathematical instruction depends, in the long view, on what happens to these teachers—and there are many of them. They might, perhaps, be considered as teachers of "demonstrated promise" rather than teachers of "demonstrated competence." An institute program planned in terms of their needs could develop a greater measure of mathematical competence, stir their imagination, extend their horizons, and actually affect classroom practice to a larger extent than in the case of those of greater achievement. They also would likewise derive great profit from hobnobbing with others in the social atmosphere which institutes provide, and I covet for such teachers this kind of opportunity, which many of them desire.

I know that Foundation funds devoted to the improvement of mathematics instruction are limited and I know that financial support cannot be provided for all projects of recognized need. I wonder, however, if there is any investment which, in the long run, will yield greater return than that which is used to improve the teaching of arithmetic and to study its role in the evolution and development of major mathematical concepts. The splendid conferences which, from time to time, deal with this topic and National Council programs dedicated to the same end touch only a very small segment of the teachers involved. However, the quality of instruction in arithmetic can exercise a very large influence on choices to be made in later years when mathematics becomes an elective and students of large potential ability are rejecting courses in mathematics on the high school level because "it never did make any sense to me." It is in the elementary school that students are first introduced in any formal manner to mathematical concepts, and what happens later depends to a very large extent on the nature of this initial introduction. Any program responsible for building conceptual insights and understandings in arithemtic will, in the long view, strengthen the very fiber of all mathematics instruction.

The educational level reached by a free people depends in great measure on the quality of instruction in both the elementary and secondary schools. Great teaching will tend to develop great citizens, and any thoughtful effort to raise the level of teaching effectiveness in the public schools of America commands our highest respect. But there is no easy or well defined road to follow in the development of great teachers. Competence in the subject-matter field involved is universally recognized as one essential factor, but while "subject-matter competence" is a necessary condition to teaching power it is not a sufficient condition. Great teachers are recognized for their ability to guide all students toward the achievement of their highest possibilities and to release the potential greatness which resides in each of them. If such desirable ends are achieved in mathematics, for example, large mathematical insights are indeed needed, but so are large human insights. Each of these factors is found in the really great mathematics teachers and neither of them should be neglected in any program designed to guide teachers of mathematics toward the achievement of their own highest potentialities.

Reviews and evaluations

Edited by Richard D. Crumley, Iowa State Teachers College, Cedar Falls, Iowa

BOOKS

Arithmetic at Work, Book 1, Howard F. Fehr and Veryl Schult, Boston, D. C. Heath and Company, 1955. Cloth, v+457 pages, $2.80.

Arithmetic in Life, Book 2, Howard F. Fehr and Veryl Schult, Boston, D. C. Heath and Company, 1956. Cloth, v+422 pages, $3.00.

Junior high mathematics has an uneasy stability in its curriculum. Arithmetic crowds up from below; algebra and geometry down from above; and the proportions of each in the seventh and eighth grades are unsettled in many classrooms and textbooks. These new textbooks by Fehr and Schult find a compromise which combines the best and most appropriate topics from these fields. There is much review of arithmetic from the fourth, fifth, and sixth grades, as there should be. Moreover, there is an excellent balance throughout the books between skills and information.

The methods of explanation used and of teaching suggested are both practical and varied; for example, there is a consistent inductive approach through pupil experiments. It is now up to the teachers who use these books actually to have the pupils do this work, not skip it. There could, with profit, be even more of this material than there is, especially in the arithmetic and algebraic sections. It is best in the geometry sections, where it is easier. A few explanations become mechanical rather than rational; for example, the description of long division (1:65).[1] There is an amount of repetition between the two books which may create an impression in the minds of teachers and pupils that the junior high school is a place where we mark time. Much is inevitable review, and certainly it would be dangerous to feel that such a subject as "Per cent" need not be explained in the eighth grade since it was taught in the seventh grade and so the pupils know all about it. On the other hand, need we repeat such topics and pictures as that of "Government Bonds"?

Writing textbooks "far from the madding crowd" would be just as foolish as feeling that you should teach topics to a class without regard to their present interests and abilities. Progress must be made slowly from what teachers are now doing to new topics and methods. It is disappointing, though, not to find these particular books making a greater contribution to progress by stretching the teachers and pupils a little more. One has the feeling that the authors have held back on their ideas and originality.

Let us examine several of the more mechanical aspects of the books, and then discuss the treatment of arithmetic, algebra, and geometry.

There are chapter summaries and word lists which can be put to very good use. At the beginning there is a distinction made in these summaries between "relations" and "facts" which is not at all clear, and which is soon dropped by the authors. Some of the difficulties of making clear, short statements in such summaries will be discussed below under the subject headings.

These books were published during what will probably be looked back on as the "color epidemic." There are places where the color in the book is both attractive and useful, but in general it is neither. When color is used in constructions to make clear the distinction between the problem and auxiliary lines, this is good; when a blob of solid color helps to define an area under consideration, then it is justified. Irregular spots of it on top of drawings or diagrams often leave the impression that it had to be used every so often since it was available, whether it made sense or not; coloring people's hands, or filling in the backgrounds of illustrations (1: 14) is sometimes distracting. Nor are the colors used very attractive shades—but this is a matter of taste. The most serious criticism is the extremely poor register of the printing between the color and the black plate. Usually it is just confusing, sometimes mathematically inaccurate. In one example, an illustration to help read angles was off 1° in 10° (1: 230); another example using color to indicate the center of circles was off $\frac{1}{32}''$ in $\frac{1}{4}''$, an amount which is very obvious (2: 339). There are many other examples of poor register that are detrimental to understanding. This may have been only on the review copy since others were not examined, but it indicates poor control of the printing.

Unfortunately there are other examples of inferior production so that the physical book does not equal the value of the writing and the editing. The total impression from an art standpoint is amateurish, although Book 2 is generally better than Book 1. In addition to the belabored use of color, there is a certain fussiness and confusion in the layout of the pictures, which is probably supposed to lead to an informal, relaxed atmosphere and often does. Other times it is just plain wasted space (1: 76–77; 1: 104; 2: 284–285). The appearance of other pages

[1] This means the first book, page 65.

with few illustrations is excellent, however. They seem open and easy to read. The outside of the book is colorful but confused. Such comments are pure opinion and may not be shared by other people.

The connection between the picture (1: 160) and the mathematics (1: 189) is often very tenuous and obscure. Some labeling of illustrations could be improved; for example, the word "circle" appears well inside the figure and tends to suggest the area, rather than the curve (1: 91). Should a parallelogram have rounded corners (1: 5), or is it not supposed to be a parallelogram? A couple of pages printed through so that the words appeared on the back. Some photography or half-tone reproduction is without snap. Sometimes the planning leads to the part of the picture that one would like to look at being bound into the binding (1: 342–43).

Arithmetic is well presented and filled with interesting examples. For instance, there is excellent material on estimating (1: 12); and good, inductive preparation for the rule of moving the decimal point in division (1: 180). On the other hand, here are four typical questions about the presentation: (1) "A ruler with sixteenth-inch marks measures no more accurately than to the nearest sixteenth of an inch" (1: 9). Is it never possible to estimate between the marks? (2) "If the ending side [of an angle] does not pass through a degree mark . . . this is called *approximate* measuring" (1: 27). Is not *all* measuring approximate? (3) Is not 3.50 a "decimal fraction" or is that reserved for numbers less than one? Why? (4) In the second book, are we to take the attitude that decimal fractions have never been heard of before? (2, 60–61). These examples show how difficult it is to protect mathematical accuracy and still keep the explanations short and simple.

The material on geometry is about the best in the book. For a good example of this, see the chapter in the second book on "Measurement of Solids." It shows the excellent integration between plane and solid geometry found throughout the books, the use of inductive approaches, of concrete materials, of helpful uses of color, of appropriate illustrations, of plenty of problem material, and good chapter review material. Here are a few questions that come to mind while reading other sections on geometry: (1) "A part of a straight line is called a *segment*" (1: 4). Any part? (2) "A horizontal line is a level line" (1: 17). Does this define in terms of a simpler idea? (3) "A line that starts from a point and goes on and on is called a ray" (1: 18). Excellent idea to use *ray* to define *angle:* but is this a good definition of a ray? (4) "There can be only four right angles around a given point" (1: 20). Should we qualify this by demanding that they be adjacent angles? (5) "If only two sides of a quadrilateral are parallel, it is a trapezoid" (1: 88). Why go back to this definition when recent trends have been to use a definition which allows a parallelogram to be a special type of trapezoid? (6) Are the faces of a rectangular solid rectangles or are they enclosed by rectangles? (1: 94). (7) "A sphere is a solid with every point on the surface the same distance from the center" (1: 97). Is a sphere a solid or a surface? Let it be admitted that all these are very fussy, even picayune questions, but they indicate the care with which authors have to prepare material for teachers who take a textbook as authority.

Algebra begins in the eighth grade book which is earlier and in more detail than some teachers are used to. There is plenty of arithmetic and review for those teachers who feel this is too early for that subject; others will welcome the chance to include such a beginning with simple equations and signed numbers. The treatment is attractive and features such original devices as diagrams to help solve equations (2: 350).

Altogether these books are more solidly scholarly than the usual junior high books, more conscious of what actually goes on in the classroom, and show enough pioneering in ideas to justify their publication. Courses based on them have a better than good chance of being successful.—*Henry W. Syer, Boston University, Boston, Massachusetts.*

Electronic Computers, edited by T. E. Ivall, New York, Philosophical Library, 1956. Cloth, viii+167 pp., $10.00.

The book under review consists of material originally published in *Wireless World*. Mr. Ivall and five other contributors are listed. The importance of automation and automatic electronic computers has been well established in the past few years but there have been difficulties in obtaining adequate descriptions on the varieties of levels represented by the public. This particular book is addressed to a very small portion of all those who are interested in computers at the non-participating level. If the reader has been exposed to a certain amount of electronics, including magnetism, and already has an appreciation of the capacity of electronic calculators, *Electronic Computers* would enhance his knowledge. On the other hand, if the reader is not conversant with some aspects of electronics as well as with solutions of differential equations, he will find the book difficult to read with any appreciation despite its non-technical appearance.

To teachers of mathematics in high schools, in particular, the book is likely to be disappointing. The mathematical content of the book from that standpoint reduces to a discussion of binary arithmetic and a not particularly adequate discussion of that.

The book is primarily an engineers' book for engineers. In the reviewer's opinion, while it skims through material on analogue and digital computers, it displays a lack of appreciation of the computers of either kind.

The book is written in England for the home trade and it bears various earmarks of provincialism, which may be considered lightly as an often needed counterbalance to American pro-

vincialism or it may be considered more seriously as misleading. Charles Babbage, for example, is mentioned as a forerunner of modern computing, but it seems never to be established that his failure to produce a machine was an inspiration to later successes. The statement made, that the hand calculators cannot be really considered as true precursors of modern electronic digital computers, the reviewer considers false. Only by increasing the appreciation of the power of calculation through better hand equipment was the proper social situation achieved in which electronic calculators could be developed.

In short, the book is addressed to a very limited audience and to very few in America will it seem worth the cost in comparison with other books in the field.—*Preston C. Hammer, University of Wisconsin, Madison, Wisconsin.*

A First Course in Algebra, revised edition, N. J. Lennes, J. W. Maucker, and John J. Kinsella, New York, The Macmillan Company, 1957. Cloth, xviii+558 pp., $3.48.

A Second Course in Algebra, revised edition, N. J. Lennes, J. W. Maucker, and John J. Kinsella, New York, The Macmillan Company, 1957. Cloth, xix+476 pp., $3.80.

For the purposes of this review these books will be referred to as First Course and Second Course respectively. Before proceeding with a detailed analysis of the strong and weak points of these texts we shall summarize our findings briefly.

These books compare favorably with the best available texts. They are attractive and usable. Much enrichment material is included in certain "C" sections for capable students. Provision for individual differences is adequate. Written problem lists are excellent. Good cumulative review tests are provided. The arrangement of topics seems conventional. The sections on graphing are especially strong. The treatment of advanced topics, determinants, progressions, trigonometry, the binomial theorem, etc., is outstanding. But the thing that impresses these reviewers most favorably is the effort to make evident the axiomatic, deductive nature of algebra. The authors are obviously trying to meet present-day criticisms of algebra texts for ignoring the fundamental postulates of algebra. Reference is made to the commutative, associative, and distributive laws. Some applications of these laws are made. One even finds some essentially correct proofs of such theorems as: *if* a *and* b *are positive integers and* a>b *then* a+(−b)=a−b. To be sure, these references to the logical nature of algebra are found almost altogether in the enrichment "C" sections. Most teachers will ignore them. But their presence is in itself a hopeful sign.

The authors are also to be congratulated for their careful treatment of such topics as the enlargement of our number system to include negative numbers and irrationals. They describe very well the extension of exponential notation to include zero, fractions, and negative numbers as exponents. A logically correct treatment of this is rarely found in a high school text.

These are good books *in comparison with other available texts*. But still, many criticisms can be levied against them. They are rule bound. The familiar rules instructing the student upon every algebraic operation sometimes border upon the ridiculous. Students may spend time memorizing instead of thinking. Although the authors recognize the necessity for emphasizing the logical nature of their subject, yet they have not infused their writing with the spirit of deductive logical reasoning. In several places correct conclusions are reached by invalid reasoning. Often when it would be possible to formulate clear, precise definitions for mathematical concepts these are withheld. Much meaningless language is used. Now and then the terminology conflicts with that favored by mathematicians. Key concepts like *variable, function, constant*, etc., are presented in a traditional manner, unsatisfactory from the standpoint of modern mathematics. The grievous error of confusing *number concepts* with the *symbols* we write to refer to these concepts arises again and again to muddy the exposition.

We call attention to a few specific features of First Course which seem good. The introduction of signed numbers is skillfully done. An interesting discussion of *inventing numbers* occurs on page 86. Teachers who like to present equations from the *balancing* point of view will enjoy the discussion on page 94. A careful distinction is made between the use of the sign "−" to indicate subtraction and its use to indicate a negative number. Chapter 13 discusses the graphs of linear equations with emphasis upon the important concepts, *slope* and *intercept*. Irrational numbers are presented quite well in chapter 15. The student who uses this text should learn that the symbol $\sqrt{4}$ stands for 2, not for ± 2.

We point out the following objections to the handling of certain topics in First Course. Basic algebraic concepts are introduced through study of the formula rather than the equation. Possibly most teachers prefer this approach, stressing the application aspect. We prefer the equation for certain fundamental reasons. The distributive law, $a(b+c)=ab+ac$, is noted but is not used to develop the theory of multiplication, division and factoring. We encounter a computation like $2a+3a=5a$ even before this important law is formulated. It is not made clear that this result depends upon the distributive law, since $2a+3a=(2+3)a$. Later in the text, multiplication of polynomials is presented in vertical form. It would be better to re-emphasize the distributive law at this point, writing

$$(x+2)(2x+3)=x(2x+3)+2(2x+3), \text{ etc.}$$

The formal presentation on page 195 of an algorithm for dividing polynomials seems psychologically unsound. It would seem better simply to define division of polynomials in terms of multiplication and then set problems before the students.

On page 240 we find an incorrect definition

of prime number. The usual definition is that a prime number is a *positive* integer *greater than* 1 having only itself and 1 as positive integral divisors. On page 242 reference is made to *the largest monomial*. This language is unfortunate. We compare the *degrees* of monomials. Factoring is not treated carefully. One can not speak sensibly of factoring until the coefficient field has been specified. On page 299 the number 2 is called a non-fraction. This is peculiar language. It is said on page 306 that the expressions $-2+x$ and $x-2$ *have the same value*. This is without meaning. What is this common value that these two polynomials possess? A ratio is defined on page 329 as an *indicated* quotient. What purpose does the word *indicated* serve other than to confuse the concept of what a ratio *is* with how we *write* a symbol for a ratio? The *number* 4 is called a *constant* because it *does not change*. Mathematicians would say that the *symbol* "4" is a *constant* because it refers to a *unique number*. In chapter 14 reference is made to *a simultaneous equation*. We speak of a *pair* of simultaneous equations. Simultaneity is a *relation* between equations, not a *property* of one equation. This recalls the story of the kindly old lady who said, "Oh, how alike your twins look, especially the boy." A logical *non sequitur* occurs on page 415 when it is remarked, *the decimal expansion of* $\sqrt{5}$ *does not lead to an exact result, thus* $\sqrt{5}$ *can not be expressed as a rational number*. Neither does $\frac{1}{3}$ lead to a terminating decimal.

In discussing quadratic equations it is said that *if* $(x-2)^2=25$, *then* $x=7$ *or* $x=-3$. Later the language, *if* $x(x+5)=24$ *then* $x=-8$ *and* $x=3$, is used. The *or* terminology would seem better in this context. Presentation of the quadratic formula in First Course seems undesirable. Let students use the *completing-the-square* technique long before mentioning any formula.

In Second Course one finds grouped on pages 20 and 21 the important basic postulates of algebra. Even the *uniqueness* of sum, product, etc., is assumed. It is pointed out carefully that the oft-stated axiom, *if equals are added to equals the sums are equal*, is no more than an assertion of the uniqueness of the sum of two numbers. This is quite correct. We know of no other high school algebra text which makes this fact clear. These reviewers have often asked students to give a numerical example of this famous axiom: *if* $a=b$ *and* $c=d$ *then* $a+c=b+d$. We then write down, *if* $3=3$ *and* $4=4$ *then* $3+4=3+4$. When students see clearly that the only assertion being made is that the sum of 3 and 4 is unique, they usually feel that their earlier mathematics teachers have deceived them.

An interesting section on pages 82 and 83 indicates how the axioms can be used to establish rules for computing with signed numbers. There is a fascinating application of graphs to the solution of work and mixture problems on page 186. In a "C" section on page 224 the familiar algorithm for square root extraction is related to the formula $(a+b)^2=a^2+2ab+b^2$.

When complex numbers are introduced, the important observation that *they can not be linearly ordered* is made. The chapter on logarithms keeps before the students the basic meaning of logarithms by presenting many solutions in exponential form as:

$$\begin{aligned} 18 &= 10^{1.2553} \\ 34 &= 10^{1.5315} \\ 18\times 34 &= 10^{2.7868} \\ &= 612. \end{aligned}$$

The work on trigonometry is elaborate. The sine and cosine laws are derived and used. A few fundamental identities are developed. If this material were taught thoroughly, then the regular trigonometry course could emphasize graphs, identities, trigonometric equations, etc.

We re-emphasize that Second Course is an outstandingly good text in that it presents clearly the axiomatic foundations of algebra and actually shows how these axioms may be used. It contains plenty of good written problems. There is much challenging material for good students. Chapters on graphing are very good and should be covered thoroughly by the teacher. Advanced topics are treated with competence.

Of course many specific complaints can be levied against the book. The axioms are collected in separate sections. They are not really used throughout the body of the text. On pages 82 and 83 the proofs of the rules for calculating with signed numbers are not complete. For example, from the fact that $ab+(-a)b=0$ it is inferred that $(-a)b=-(ab)$. This does not follow automatically from the axioms listed on page 81. We need to know that every number "a" has a *unique* negative $-a$ in order to conclude this.

Again in Second Course the powerful distributive law is not utilized in developing skills in multiplication, division, and factoring. It is implied by the great bulk of examples that polynomials are to be factored over the ring of integers and then a few examples involving fractions are tossed in without comment.

A discussion of the remainder and factor theorems on pages 119 to 123 uses functional notation. This has not been explained in the text. This is enrichment material, of course, but the teacher must explain the notation carefully if it is to be understood.

On page 132 a reference to *integral algebraic expression* is made. *Polynomial* would be a better term. A discussion of graphing states that the graph of $y=3/2x-2$ *intercepts* the y-axis at -2. It would seem better to employ the usual terminology *intersects*, using intercept only as a noun.

The discussion of fractional exponents on pages 231 ff. is sound. It would be well to point out that if $a>0$ we are free to define $a^{1/2}$ to be either $\sqrt{a}$ or $-\sqrt{a}$. The former choice is more natural. It also would be well to observe that we do not define a^0 if $a=0$.

An incorrect remark relating to complex numbers occurs on page 238. It is said that *the expression* $3+2i$ *is not a sum in the sense that the parts* 3 *and* 2i *can be added*. There seems to be confusion here between *numbers* and the *symbols* we write for numbers. Certainly both 3 and

$2i$ are complex numbers *and* their unique sum exists. This sum is the number $3+2i$. This is just like saying the sum of 3 and 2 is $3+2$. Of course addition of complex numbers differs from addition of real numbers, just as addition of real numbers differs from addition of fractions and addition of fractions in turn differs from addition of integers. The authors probably have in mind the fact that whereas we have a simple symbol "5" to represent the sum of 3 and 2 we have no such simple symbol to represent the sum of 3 and $2i$.

The remark on page 263 that *if* b^2-4ac *is a perfect square the roots of the quadratic are rational* is valid only if the coefficient field is restricted to rational numbers.

We conclude by remarking that these texts contain far fewer errors than most. The attempt to call attention to the deductive aspects of algebra is a gallant one. However it is definitely not satisfactory. The high school text which will develop algebra logically in a manner acceptable to modern mathematicians has yet to be written.—*Charles Brumfiel, Ball State Teachers College, Muncie, Indiana, and Robert Eicholz, Burris Laboratory School, Muncie, Indiana.*

Fun with Figures, J. A. Hunter, New York, Oxford University Press, 1956. Cloth, xi+160 pp., \$3.00.

Fun with Figures contains a collection of problems that are concerned with common, everyday events with which most persons are familiar.

Each problem is separate from, and unrelated to, the other problems in the collection. Each problem has an interesting title and is couched in informal and interesting language.

For example, Problem 28, The Beauties of Kalota, tells us about a strange custom that a woman of Kalota must never make two consecutive true or untrue statements. Because of this custom, the problem of discovering the ages of four beautiful sisters depends upon an analysis based upon mathematical logic. A birthday, a telephone number, a license plate, a card game, are a few of the ordinary quantitative situations that become the sources of intriguing mathematical secrets.

The solutions of the problems involve elementary algebra and logic. Some of the problems may be solved by the methods of Boolean Algebra. Several typical solutions are included in the book as well as the answers for all the problems.

This book will provide a valuable source of enrichment material for both first and second year algebra students. Junior and senior high school mathematics teachers will find this book to be a valuable aid for creating and maintaining the mathematical interest of their pupils, as well as providing practice in solving interesting problems. Both junior and senior high school libraries should have several copies of this book available for the students.

Fun with Figures offers real interest, fun, and help for both students and teachers of secondary mathematics.—*Dale Carpenter, City Board of Education, Los Angeles, California.*

Introduction to Statistical Analysis, Second Edition, Wilfrid J. Dixon and Frank J. Massey, Jr., McGraw-Hill Book Company, Inc., 1957. xiii+488 pp., \$6.00.

This book is written for a basic course in statistics to be taken by students from all fields in which statistics find application. The only mathematical ability assumed of the reader is a knowledge of elementary algebra.

The topics considered are probability, sample and population characteristics, estimates of the mean and variance, tests of statistical hypotheses and their power, analysis of variance and covariance, regression and correlation, *chi* square tests, sequential analysis, and nonparametric statistics. Many substitutes for classical statistics have been suggested. There are discussion questions, class exercises, and problems at the end of each chapter, and a good collection of references at the end of the book. The thirty-three tables, covering 104 pages, include percentage points of the distributions of a number of "new" statistics as well as those for the normal, *chi* square, t, and F variables. The power curves of t and F tests also are listed.

The authors have done a very commendable job, as the book stresses statistical inference rather than the purely descriptive aspects of statistics. They wisely emphasize the fact that the hypothesis and its test must be stated before the sample items are observed. The importance of considering the probability of the type-two error while designing a test is demonstrated many times.

Since the authors, like many writers of books on this level, prefer not to introduce the concept of mathematical expectation, they have trouble defining the population mean and variance. The reviewer also feels that their presentation would have been more complete if they had listed the alternative hypothesis as well as the null hypothesis with each example.

Teachers of general introductory courses in statistics should give some consideration to this competent piece of work.—*Robert V. Hogg, University of Iowa, Iowa City, Iowa.*

Modern Trigonometry, William A. Rutledge and John A. Pond, Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1956. Cloth, xi+243 pages, \$3.95.

This text presents trigonometry as a study of the six trigonometric functions. Although it contains the usual topics, the emphasis is on "analytic trigonometry." There are several features of the order and type of exposition which deserve special attention. These are: an early introduction to the notion of general angle (chapter I); a co-ordinate plane and a distance formula as basic tools in the exposition; a treatment of the trigonometric functions prior to the trigonometry of a triangle; a presentation re-

quiring the use of algebraic skills but not neglecting geometric imagery; a development of special topics in terms of basic trigonometric relations; an exposition in the form of "proof" for some statements.

The introductory section on "co-ordinate systems" seems too brief for its importance as a tool in the subsequent work of the text. One would like to have this section include material related to the graphs of equations which play key roles later. This material would have complemented the sections dealing with the notion of function.

The style and techniques of the text would recommend it as a college text. At the same time they would make its use in most high school classes rather difficult.

The text is on good quality paper and attractively bound. It contains diagrams appropriate to the content. The typography is such that terms used in a technical sense for the first time are indicated by bold-face type. Other key words, phrases, and "frequently-referred-to" statements are set in italics.

There is an adequate supply of exercises of routine nature. A few exercises are intended to stimulate self-discovery. There are "oral exercises" to aid in checking the student's grasp of the ideas. Answers to odd-numbered exercises and the usual tables are included. The only typographical errors noted were on pages 10 and 131. —*Robert C. Seber, Western Michigan College, Kalamazoo, Michigan.*

Topics in Number Theory, Volume I, William Judson LeVeque, Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1956. Cloth, x+198 pp., $5.50.

Topics in Number Theory, Volume II, William Judson LeVeque, Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1956. Cloth, viii+270 pp., $6.50.

The author opens his first volume with an introductory chapter attempting to communicate to the neophyte the flavor of number theory, by means of a statement of the kinds of problems which occur in the subject and some discussion of the types of proof required. Here he faces what is both the charm and the source of difficulty in number theory: diverse ingenious methods of proof—a phenomenon slightly akin to prospecting for gold, where a small operator may strike it rich, though in these days heavy machinery deftly used produces more consistent results. The author makes the point that with practice "tricks become methods" (as is true in all mathematics and elsewhere). One of the strong attributes of the book is smoothing and shortening the path from trick to method, and cultivating in the reader a feeling for the problems and methods of number theory.

The next four chapters deal with topics usual to a first course in the subject (Euclidean algorithm, congruences, primitive roots and indices, quadratic residues). Here the unusual details are discussion of the equivalence relation, a more complete treatment than is usual of a single linear congruence in several unknowns and conditions for the solutions of a set of congruences $x \equiv c_i \pmod{m_i}$. The reviewer is a little disappointed at the notation (a/b) instead of $(a|b)$ for the Legendre symbol.

Chapter 6 is concerned with certain number-theoretic functions: Möbius function, greatest integer in, sum of divisors, number of divisors, and $\pi(x)$, the number of primes less than or equal to x with some elementary properties of $\pi(x)$ including Erdös' elegant proof of Bertrand's conjecture that between n and $2n$ $(n>1)$ there is always at least one prime number.

Chapter 7 contains the usual results about the sum of two squares and the proof of A. Brauer and R. L. Reynolds that every positive integer is the sum of the squares of four integers.

In chapter 8, the discussion of the Pell equation and the general quadratic diophantine equation is more complete than is usual in texts, and the Farey series is used to deduce Hurwitz' theorem: if α is any irrational number, then there are infinitely many integral solutions x, y of the inequality $|\alpha - x/y| < 1/\sqrt{5}y^2$.

The subject of rational approximations to real numbers is continued in the final chapter. Here continued fractions are introduced as a means of getting such approximations. Though this is an interesting point of view, it seems to the reviewer that some of the intrinsic beauty of continued fractions is thereby lost.

The second volume contains advanced topics requiring more mathematical experience: algebraic numbers, the Thue-Siegel-Roth Theorem (a deep theorem related to the Hurwitz theorem above and recently proved by Roth), the Dirichlet theorem that the set of numbers $an+b$ where n ranges over the positive integers and a and b are relatively prime contains an infinite number of primes, and the prime number theorem.

It seems to the reviewer that there might be some virtue in postponing a review until the book in question had been used in a class. Then a list of misprints could be given, and the reviewer could speak with more authority on the clarity and teachability of the text. As it is, some of my impressions might not be borne out by classroom use. Certainly the author takes great pains to inform the reader where he is going and what he is trying to do. The theorems are precisely stated and the proofs complete, but the reviewer has the impression that at times more detail would be welcomed by the average reader. For instance, in exercise 2 on page 9, the author assists the reader by remarking, "since the terms $\sum_1^n m^k$ drop out." I do not believe I would have known what was meant by this unless I had known how to solve the problem without it. On page 70, the inequality in the sixth line from the bottom should be: $qx/p + \frac{1}{2} < (q-1)/2$; or at least this inequality is true and the author's results come more easily after this alteration. On page 136, the first reference cites erroneously "the theorem that every integer can be represented as the sum of three squares."

The reader who does not wish to go far into number theory will probably want to pick and choose to a certain extent in the first volume; he will have to be prepared to use pencil and paper and grey matter as he reads—a procedure to be recommended with most collegiate textbooks. Aside from the novelties of detail mentioned above, the chief contribution of the first volume is its thorough preparation for the second volume—not only in the establishment of results, but in the training in methods and the cultivation of maturity. It is the second volume which contains the most material not found elsewhere in English textbooks and, indeed, some results and proofs not found elsewhere. This is the volume which the specialists in number theory have been waiting for, for the use of their students, and it is to be hoped that many who begin with the idea of merely paddling by the shore may be enticed into deeper waters and uncharted seas.—*Burton W. Jones, University of Colorado, Boulder, Colorado.*

INSTRUCTIONAL MATERIALS

Geniac Kit (*K*1)—Electric Brain Construction Kit. Berkeley Enterprises, Inc., 815 Washington Street, Newtonville 60, Massachusetts, $17.95 plus 10% for shipping.

Tyniac Kit (*K*3)—Electric Brain Construction Kit. Berkeley Enterprises, Inc., 815 Washington Street, Newtonville 60, Massachusetts, $9.95 plus 10% for shipping.

These two kits differ only in complexity; the more expensive has six "multiple-switches," the smaller has four. These multiple-switches are the heart of the device and are masonite discs, with a hole in the center and 64 other holes arranged four each on 16 radiuses. "Jumpers" are put through two adjacent holes in any one radius and these serve to connect, electrically, the heads of two bolts which are mounted in corresponding holes on a masonite panel which is provided. This gives 16 positions in which the switch may be left and three different positions along each radius for contacts. Since several jumpers, on the same or different radiuses, can be set up to "make" or "break" connections simultaneously, there are a great many different ways that these switching elements can be set up and connected.

Other parts of the kit include nuts and bolts, and metal and rubber washers for mounting the switches; wire, a battery, a battery-clamp, flashlight bulbs and sockets, and an on-off switch for completing the electric circuits; a screwdriver and wrench; a crayon and printed labels for identifying positions of switches; and a manual of instructions. The smaller kit has fewer parts and sometimes omits a less important item or two.

The manuals describe 33 or 13 "experiments" which can be performed on the "electric brain machine." They range from simple on-off switches, "Joe Saverelli's Rock Quarry," to such devices as a "Binary Adding Machine." Diagrams (and for the Tyniac machine, templates) are provided to simplify the work of connecting the various circuits.

The devices are fascinating, scientifically sound and potentially educational. At a high school level there are many students who will be delighted to try out these devices and learn much from them. Nevertheless, from an intellectual standpoint the kits and instructions are disappointing and—I hope I may use the word without being misunderstood—dishonest. This is the same criticism which could be aimed at a book for which Mr. Berkeley is responsible: *Giant Brains, or Machines That Think*. It is a good book but a terrible title; scientific objectivity is sacrificed for sales value. Likewise, with these kits showmanship overcomes preciseness; if you will examine the first sentence in the preceding paragraph you will find three examples of what is meant: (1) "experiments," (2) "electric brain machines," and (3) "Joe Saverelli's Rock Quarry."

First, the devices to be built are not "experiments" although they probably could be called such as much as many performed in most high school physics or chemistry laboratories. They are directions for putting equipment together to reach foregone conclusions. The whole tone of the direction manuals aims at making it possible to do the experiments in the booklet, which *someone else* has thought out. There should have been more emphasis on the general principles involved in such circuits and switching; and on the encouragement of thinking through, drawing diagrams for and constructing of *original* circuits. Only in making these is mathematical learning involved. Many pupils will do so, for the equipment is very flexible in its application. The manual for the TYNIAC leans in this direction with its "Part III: Introduction to Boolean Algebra for Circuits and Switching."

Second, calling these contrivances "electric brain machines" is overselling, regardless of how fascinating the title sounds. Once someone discovers how fundamentally simple the ideas are, he will be let down and disappointed for having been told they are "brains." Such analogies can be used by a teacher who will safeguard the misunderstandings with qualifications.

Third, the humorous titles in the devices are good teaching, but need to be followed up by an analysis of the generalities underlying the device. As the material is presented here, it seldom goes beyond the "puzzle" stage. For purposes of amusement this is fine; educationally a pupil or teacher needs more.

These kits would help the teaching of mathematics greatly if used wisely. The motivation in having pupils make, display, and *explain* these devices to a class is invaluable. It is worthwhile to buy the whole kit rather than just the manual of instructions; enough parts are specialized in design to justify this rather than trying to buy them locally. In the classroom always remember the necessity of using such devices

to illustrate and stimulate thinking, not to counterfeit it by following instructions.—*Henry W. Syer, Boston University, Boston, Massachusetts.*

TESTS

Madden-Peak Arithmetic Computation Test, Richard Madden, San Diego State College, and Philip Peak, Indiana University. World Book Company, Yonkers-on-Hudson, New York. Test Booklet $3.00 per 35, net; Answer Sheet $1.15 per 35, net; Machine Key $0.20 each, net; Specimen Set $0.35 each, ppd.

This test is one of eight mathematics tests in the Evaluation and Adjustment Series. The purpose of this particular test is to measure the skills needed in performing the basic operations of arithmetic. It is intended chiefly for use in the high school and adult levels, although a limited standardization has been done at the junior high level. The test yields separate scores on the following parts: Addition and Subtraction; Multiplication and Division; Common fractions; Decimal fractions, Mixed Decimals and Per Cents; Mental Computation and Estimation.

The test is organized so that it can be administered in a 55-minute class period with a total working time of 49 minutes. Answers are marked in space provided in the test booklet unless separate answer sheets are used to permit machine scoring or quicker hand scoring.

A system for interpreting scores simplifies the use of the test results. Test scores may be interpreted in terms of (1) percentile ranks, by grade for total score and for each part score, (2) stanine levels, by grade for total score. Standard scores for individuals and classes enable comparison with national mid-year norms, comparison with achievement in arithmetic computation, comparison with general mental ability, and analysis of group strength and weaknesses as a guide for improving the instructional program.

The reviewer gave this test to a group of ninth-grade students and found the test to be very easy to administer and score. This test can be a very useful measuring device if the teacher interprets the results as measures of computational skill and not as a measure of the total arithmetic program.

The section on mental computation and estimation is the weakest part of the test. It is composed of sixteen examples, many of which are similar to those in other parts of the test. Since the student does not use pencil and paper, it is assumed that the choice of one of five possible answers will indicate success in mental computation and estimation. It seems to the reviewer that this is a large assumption.

In general the test is very well done and no doubt is a contribution to the analysis of computational success in arithmetic.—*George E. Immerzeel, Iowa State Teachers College, Cedar Falls, Iowa.*

I shall assume that the all-important goal of the government and citizens of the United States is to produce and maintain a situation where all our people can live in freedom, in safety, in comfort, in happiness and in peace. I assume we have no desire to conquer other nations—but neither do we have any intention of letting anyone conquer us. However, we have learned from bitter experience that we cannot guarantee our safety by standing alone in the world. Indeed, we must ally ourselves firmly with other free nations to resist the spread of slavery and tyranny.

Freedom—comfort—safety—a good life. Those are the things we strive for. Those are the things our advancing technology seeks to achieve. And before we get too excited about what other nations are doing—or whether they are ahead of us—let us ask whether we are moving forward as fast as we should toward our goals and then ask whether or how the things the other fellow is doing may impede our progress.—*Taken from Dr. Lee A. DuBridge, "The Crisis in American Education,"* Age of Science, *Vol.* 2, *December*, 1956.

Edited by Robert S. Fouch, Florida State University, and Robert Kalin, Florida State University, Tallahassee, Florida

Testing the understanding of the concept of equation

by Eugene D. Nichols, Florida State University, Tallahassee, Florida

There are basically two types of tests used by teachers of mathematics: objective and subjective. In objective tests, the judgment of the scorer does not affect the score; the objective test scored by a number of persons yields the same score on each scoring. Different scores on subjective tests frequently result not only when scored by different persons, but even when scored by the same person at different times.

It is not the purpose of this brief discourse to discuss the frequently mentioned inadequacies as well as strengths of each type of test. Rather, the purpose is to point out some possibilities for better construction and more effective use of tests. I shall limit myself to remarks concerning the objective tests administered in written form. Then I will make some suggestions as to how more insight into the nature of a student's knowledge may be obtained through oral work.

It is a truism to state that tests are an essential part of teaching. They have the function of judging the effectiveness of teaching done, as well as having implications for the teaching to be done. Through testing, however, the teacher not only must obtain an estimate of the quality of his teaching and the quality of the student's learning, but he also must develop an insight into ways of judging the quality of the test itself. That is, he must develop the sense, somewhat as the artist does, by which he will be able to interpret the results of the test so as to use them for improving the quality of his instruction, the amount of learning on the part of students, and the effectiveness of the instrument with which he chose to test the two.

One of the ideas that is frequently advocated is that the end result of teaching should be understanding rather than just an ability to perform mechanical manipulations. Such a belief has more implications than many who hold it realize. Certainly it has implications for emphases in teaching, as well as in testing. A teacher is inconsistent if he insists that he teaches for understanding, while his tests call exclusively for the display of ability to apply rules. Inasmuch as understanding should be an outcome of learning, the teacher must, in his teaching and subsequently in the testing, get at the measure of understanding the student derives from the instruction.

In constructing a test, it is imperative to identify the important concepts in an area of a given subject matter chosen for instruction. Let us assume that the area is the solution of linear equations ordinarily taught in first-year high school algebra. Before teaching a unit on linear equations, the teacher decides upon the

important ideas which he will want his students to acquire as a result of instruction.[1]

Examples of some of these ideas stated in the form of questions may be as follows:

1. What is an equation?
2. What is a root of an equation?
3. What are equivalent equations?
4. Must an equation have only one root?
   May it have more than one root?
   May it have no roots?

One must observe that many of the questions above presuppose the ability to solve equations. It is assumed that the teacher, in constructing the test, included a variety of items through which he obtains a measure of the extent to which the students do possess this ability. Furthermore, the teacher must attempt to establish to what extent the inability of the student to solve certain equations accounts for the lack of understanding of some of the important concepts.

Next, the teacher may have decided that the student has the concept if he is able to provide a response to a question which he would not be able to provide had he not had the understanding of the given concept. Thus, the possession of the concept is ascertained by inference. The teacher then will proceed to construct items designed to test the concepts he set out to teach. A few sample items which he might use may be as follows: (The items are, of course, preceded by a set of directions telling the student to mark only one response; in case none of the first four choices is correct the student is to mark 'none of these'.)[2]

1. Which of the following is *not* an equation?
   a) $3-x=5$
   b) $\frac{7}{8}=2$
   c) $7\times4=28$
   d) $N=N-1$
   e) none of these

The correct response to this item is 'none of these,' provided the student was taught that an equation is recognized by its form—an equality sign with expressions on both sides containing numerals or variables (or both). He must have been led to recognize that whether an equation is a true statement, a false statement, or an expression which is neither true nor false is irrelevant to the problem of deciding whether it is an equation or not. Thus, '$\frac{7}{8}=2$' is an equation, in spite of the fact that, upon further reflection, one ascertains that it is a false statement. Similarly, '$3-x=5$' is an equation, although no truth value can be attached to this expression until 'x' is replaced by a numeral.

2. The root of the equation '$3x=5x$' is
   a) $\frac{3}{5}$
   b) $\frac{5}{3}$
   c) 0
   d) 1
   e) none of these

The correct response is '0'. Here the student ought to recognize that the root of an equation is a number whose name, put in place of the variable, will result in a true statement.

3. Which of the following equations is equivalent to the equation '$x/x=1$'?
   a) $x=x$
   b) $\dfrac{2x}{2x}=1$
   c) $x=1-x$
   d) $x=1+x$
   e) $x=\dfrac{1}{x}$

The correct response is '$2x/2x=1$'. Here the student must recognize that equivalent equations have exactly the same roots. Every number, except 0, is a root of the equation '$x/x=1$'; the same is

[1] Some of the ideas expressed in this article are derived from my experience while a member of the University of Illinois Committee on School Mathematics. Dr. Max Beberman is presently the Director of the Mathematics Project of this Committee.

[2] I shall use single quotation marks to indicate that the referent is a name of a thing, rather than the thing itself.

true for the equation '$2x/2x=1$'. It is true for no other equation given in the responses above. (Note that 0 is a root of the equation '$x=x$'.)

4. Which of the following equations has more than one root?

a) $5n=0$

b) $\dfrac{-6}{n}=0$

c) $|n-1|=5$

d) $|3-n|=-3$

e) none of these

The correct response is '$|n-1|=5$'. The roots are 6 and -4.

It is doubtful that a teacher may get significant insight into individual students' thought processes and their extent of understanding of the sample ideas listed above on the basis of the responses to items like the above alone. Records of oral work and the analysis of this work, although time-consuming and laborious, may prove invaluable.

A teacher may choose a few students who gave the right response to an item and students who chose each of the wrong responses. Letting the student do all of his work aloud and recording his deliberations may provide the teacher with a wealth of information as to how the student arrived at his answer. It is only through an analysis of this kind that a teacher may be able to ascertain whether students choose a correct response by means of correct thought patterns, and whether they arrive at wrong responses by means of incorrect thought patterns.

To illustrate some kinds of knowledge that a student may evidence through oral testing, I shall consider the following test item:

The intersection of the loci of which pair of equations is an empty set?

a) $2x-y=5$
$2y=4x-7$

b) $3x+5=y$
$y-2x=1$

c) $2x=5$
$y=-3$

d) $x+y=2$
$x-y=1$

e) none of these

First, the tester should obtain some evidence that the student understands that 'locus of an equation' is synonymous with 'graph of an equation.'

Second, the student will need to think of a graph of an equation as being a set of points whose co-ordinates satisfy the given equation. Thus, the roots of the equation are numbers which are the co-ordinates of points; the points constitute the locus in question.

Third, the student must recognize that each of the equations given in the question above has a straight line for its locus. Furthermore, in a plane two straight lines either intersect or they are parallel.

Fourth, the student must manifest understanding of "intersection of sets" and "empty set." That is, the tester will seek evidence that in choosing a response to the test item, the student is searching for a pair of equations such that the graphs of these equations are parallel lines.

It should be added that the evidence of understanding of each of the above concepts need not be expressed in words. The tester, through keen observation of *all* of the student's behavior, infers the extent of understanding that the student probably possesses.

In summary: I have attempted in this article to express a view that, if a teacher believes that students must gain understanding of concepts as a result of teaching, then the teacher's tests must reflect this belief. I have chosen the area of linear equations to illustrate some ways by which the evidence of the understanding of the concept of linear equations can be obtained. In my deliberations, I have chosen to ignore the limitations under which a teacher usually works. At the same time, I am aware of the fact that the conditions under which many teachers work makes the type of testing suggested in this article feasible.

• **TIPS FOR BEGINNERS**

Edited by Francis G. Lankford, Jr., Longwood College, Farmville, Virginia, and Joseph N. Payne, University of Wisconsin, Madison, Wisconsin

Prime power decomposition

by Gordon D. Mock, Wisconsin High School, Madison, Wisconsin

The fact that *a number can always be expressed as a product of prime numbers* can be used to develop new understanding about equivalent fractions and about the square root of a number. Prime power decomposition is merely the language of number theory for saying that any number can be expressed as a product of prime numbers or powers of primes, e.g., $10=2\cdot 5$ and $27=3\cdot 3\cdot 3=3^3$. This idea can be used in eighth and ninth grade general mathematics.

It is necessary first to establish clearly the meaning of a prime number. A satisfactory definition is: A prime number is a number greater than one that is divisible only by itself and one. Examples are readily supplied by the students. At this age level, the Sieve of Eratosthenes for finding prime numbers can be used without fear that the youngsters will be bored. Have them write out the numbers from 2 to 100. Then circle the 2. Point out that this satisfies our definition of "prime," but that every even number from then on must *not* be prime, since it is divisible by 2. Hence, all even numbers should be crossed out. The next number not crossed out is 3. This is a prime and is circled. However, every third number from then on will have 3 as a factor and should be crossed out. Some of the numbers will already have been crossed out, but this does not disturb the process. Five, a prime number, is the next number not crossed out. Hence, circle 5 and cross out every fifth number thereafter. Continue this process, circling the next prime number and crossing out every multiple of the prime number. Thus, eventually you will have circled all the prime numbers up to 100. As a practice, you might pick at random a number that has been crossed out and determine its divisors, or its prime power decomposition.

Next it will be necessary to practice writing numbers as the product of prime numbers. The use of an exponent may be included or omitted as time permits. (It is not entirely necessary and should not be made a major problem.) Examples should be simple and not require the use of large prime numbers. For example, $8=2\cdot 2\cdot 2$; $15=3\cdot 5$; $22=11\cdot 2$. The question will arise as to how to start to "break down" a given number. Here you can review or teach the various short cuts that everyone should know about division. If the number is even, it is divisible by two; if the sum of its digits is divisible by three, the number is divisible by three; if the number ends in five or zero, it is divisible by five. After this, pupils will have to try various primes.

Assuming, now, that the pupil is able to write any "reasonable" number as a product of primes, we can discuss the reduction of fractions. Take a simple case first:

$$\frac{4}{8}=\frac{\cancel{2}\cdot\cancel{2}}{\cancel{2}\cdot\cancel{2}\cdot 2}=\frac{1}{2}$$

Point out that this system would be needless in such a simple situation as reducing

4/8, but that it might be helpful in the following:

$$\frac{21}{35}=\frac{3\cdot 7}{5\cdot 7}=\frac{3}{5} \qquad \frac{42}{70}=\frac{2\cdot 3\cdot 7}{2\cdot 5\cdot 7}=\frac{3}{5}$$

The great advantage of this system is that the fraction is always reduced as far as possible. For example, you would never leave the second example above as 21/35. Also, the fundamental idea that you are dividing, rather than cancelling, is clearer here.

In adding and subtracting fractions, it is necessary to find a common denominator, and preferably the least one. Suppose we have denominators $14=2\cdot 7$ and $21=3\cdot 7$. By inspection we can see that if 14 were multiplied by three we would have $2\cdot 3\cdot 7$ and if 21 were multiplied by two, we would have $2\cdot 3\cdot 7$ or a least common denominator of $2\cdot 3\cdot 7$. You can continue to use this inspection method or you may inductively arrive at the conclusion that you take each factor the largest number of times it appears in any one denominator.

Again, we have the advantage of removal of guesswork and we are assured that we will have the *least* common denominator.

In finding the square root of a number, prime numbers can be used again. They help to emphasize the meaning of square root. Suppose we wish to find the square root of 432. First, we note that 432 is the product of prime numbers. That is, $432=2\cdot 2\cdot 2\cdot 2\cdot 3\cdot 3\cdot 3$. Next, we recall that the square root of the square of a number is that number. It is not startling, then, to find the square root of 432 as follows:

$$432=2\cdot 2\cdot 2\cdot 2\cdot 3\cdot 3\cdot 3$$
$$=2\cdot 2 \quad 2\cdot 2 \quad 3\cdot 3\cdot \quad 3$$
$$\sqrt{432}=2\cdot 2\cdot 3\cdot \sqrt{3}$$
$$\sqrt{432}=12\sqrt{3}$$

Many square roots can be found by using this method together with a table of roots to 100. In the example above, we look up the square root of three and multiply by twelve.

Using the idea of prime power decomposition is evidently not new. Older texts of around 1900 and before included something of this idea, but contemporary writers have generally ignored it.

The method discussed above has the following advantages:

1. It eliminates mere trial and error in finding equivalent fractions.
2. It introduces an important mathematical concept with the review of arithmetic that is commonly found in the eighth grade.
3. It enlivens review sessions with something that is mathematical and at the same time interesting to most pupils.

A solid geometry student's lament

My world is full of dihedral angles,
Prisms, cylinders, and right triangles.
Millions of planes here and there
Intersecting each other in mid-air;
A sphere rolls down an inclined plane
Which is tangent to it, time at the same.
A rectangle turns around on its edge—
A cylinder of revolution with axis at edge.
Do parallel planes intersect in space?
Does not a cone have to sit on a base?
I look at my figures and I sigh—
"Is not 3.1416 the value of pi?"
Now my vision is getting hazy;
Solid geometry's driving me crazy!

—*Allene Dooley, student, College High School, Bowling Green, Kentucky.*

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Pres.—Katherine G. Engler, Memorial Junior High School, Valleystream
V. Pres.—George R. Schriro, Weldon E. Hornett High School, Farmingdale
Secy.—Emily J. Klein, Bethpage High School, Bethpage
Treas.—Louis Thymuis, Great Neck Senior High School, Great Neck

Suffolk County Mathematics Teachers Association
Pres.—Lois Bowman, Amityville Memorial High School, Amityville
V. Pres.—Jack Courtin, Northport Junior High School, Northport
Secy.—Carl R. Wampole, Hauppauge High School, Hauppage
Treas.—Jack G. Record, R. L. Simpson High School, Huntington

NORTH CAROLINA

Department of Mathematics of the North Carolina Education Association
Pres.—Lenoir William, 402 South Williams St., Goldsboro
V. Pres.—Mrs. Margaret McDevitt, Lee Edward High School, Asheville
Secy.—Lessie Cogdell, Needham Broughton High School, Raleigh

OHIO

Ohio Council of Teachers of Mathematics
Pres.—Eugene P. Smith, University High School, Ohio State University, Columbus 10
V. Pres.—Clarence Heinke, Capital University, Columbus 9
V. Pres.—Charles E. Scott, Cleveland East Technical High School, Cleveland
V. Pres.—Helen R. Brown, Steubenville High School, Steubenville
Secy.-Treas.—J. V. Naugle, Wyoming High School, Wyoming

Mathematics Club of Greater Cincinnati
Pres.—Robert D. Price, 408-C Teachers College, University of Cincinnati, Cincinnati 21
V. Pres.—Kenneth F. Scheurer, Samuel Ach Junior High, 3485 Reading Rd., Cincinnati 29
Secy.—Ethel H. McDonough, Evanston Elementary School, Dana and Trimble, Cincinnati 7
Treas.—John J. Geiger, North Avondale Elementary School, 615 Clinton Springs, Cincinnati 29

Greater Cleveland Mathematics Club
Pres.—James D. Gates, Maple Heights High School, Maple Heights
V. Pres.—James D. Bristol, Shaker Heights High School, Shaker Heights
Secy.—Evelyn M. Coates, Harding Junior High School, Lakewood
Treas.—Harold S. Miller, Roxboro Junior High School, Cleveland Heights

Greater Toledo Council of Teachers of Mathematics
Pres.—Floyd Strow, Ottawa Hills High School, Ottawa Hills
V. Pres.—Dallas Greenler, Clay High School, Toledo 5
Secy.-Treas.—Violet Davis, University of Toledo, Toledo 6

OKLAHOMA

Oklahoma Council of Teachers of Mathematics
Pres.—Mrs. Mildred G. Nesbitt, 524 N. W. 51st St., Oklahoma City
V. Pres.—Mrs. Carolyn Stewart, 2242 E. 36th St., Tulsa
V. Pres.—Mrs. Katherine W. Maddox, Rt. 2, Box 45, Norman
V. Pres.—Mrs. Alma Hopkins, Crescent
Secy.—Marie Shacklett, 1021 S. Macomb, El Reno
Treas.—Mrs. Kathryne Leverett, 5921 N.W. 54th St., Oklahoma City 12

Mathematics Council of Oklahoma City
Pres.—Norma Jones, U. S. Grant Junior-Senior High School, S.W. 48th & Pennsylvania, Oklahoma City
V. Pres.—T. L. Vickrey, John Marshall High School, N.W. 88th & University, Oklahoma City
Secy.-Treas.—Lola B. Greer, Moon Junior High, 600 N. High, Oklahoma City

Tulsa Chapter of The National Council of Teachers of Mathematics
Pres.—Joseph Whitsett, Will Rogers High School, 3909 E. 5th Pl., Tulsa
V. Pres.—Marcus Kirkpatrick, Roosevelt Junior High School, 301 N. Quanah, Tulsa
Secy.-Treas.—Mrs. Alpha Shirey, Wilson Junior High School, 1127 S. Columbia, Tulsa

ONTARIO

Ontario Association of Teachers of Mathematics and Physics
Hon. Pres.—Norman Lane, McMaster University, Hamilton
Past Pres.—H. E. Totton, Forest Hill Collegiate Institute, Toronto
Pres.—R. H. B. Cook, Harbord Collegiate Institute, 286 Harbord St., Toronto
V. Pres.—D. L. Mumford, University of Toronto, Toronto
Secy.-Treas.—Charles F. Chambers, 242 Kingswood Road, Toronto

OREGON

Oregon Council of Teachers of Mathematics
Pres.—Oscar F. Schaaf, 123 Leigh, Eugene
V. Pres.—Dana Small, Franklin High School, Portland
Secy.-Treas.—Vernon Kerley, Eugene High School, Eugene

PENNSYLVANIA

Pennsylvania Council of Teachers of Mathematics
Pres.—Joy E. Mahachek, 962 Philadelphia Street, Indiana
V. Pres.—Edward E. Bosman, Barrett Township High School, Cresco
Sec.—Freda E. Jones, Kingston High School, Kingston
Treas.—Charles M. Carroll, 505 W. Main St., Anneville
Editor—Catherine A. V. Lyons, 12 S. Fremont Ave., Pittsburgh

Western Pennsylvania Association of Teachers of Mathematics
Pres.—Melvin N. Vesely, Carrick High School, 125 Parkfield Ave., Pittsburgh 10
V. Pres.—Earle F. Myers, University of Pittsburgh, Pittsburgh 13

Secy.—Helen Malter, Coraopolis High School, Coraopolis
Treas.—Mabel Milldollar, Natrona Heights School, Freeport Road, Tarentum

Association of Teachers of Mathematics of Philadelphia and Vicinity
Pres.—Karl S. Kalman, Lincoln High School, Philadelphia 36
V. Pres.—Milton Jenkinson, South Philadelphia High School, South Philadelphia
V. Pres.—Alvesta Flanegan, Olney High School, Front and Duncannon, Philadelphia
Secy.—Sister Mary Louise Klaus, Lankenan School, 3201 West School House Lane, Philadelphia
Treas.—Mrs. Rita Kieshner, Fels Junior High School, Algon and Devereaux, Philadelphia

SOUTH DAKOTA

Mathematics Council of the South Dakota Education Association
Pres.—Kenneth Newell, Mobridge
V. Pres.—Albert Beatty, Dupree
Secy.—Marvin Simonsen, Sioux Falls
Treas.—Bill Ochs, Aberdeen

TENNESSEE

Mathematics Section, East Tennessee Education Association
Chm.—William O. Largen, 502 Willard St., Maryville
V. Chm.—Robert E. Keasling, 311 Park St., Greenville
Secy.—Mrs. Mozelle Ridgeway, 2826 N. Tunnel Blvd., Chattanooga

TEXAS

Texas Council of Teachers of Mathematics
Pres.—Margaret Kennerly, 3214 Huntingdon, Houston
1st V. Pres.—Mazelle Schulenberger, 101 Ross Ave., Cleburn
2nd V. Pres.—Arthur W. Harris, 4701 Cole, Dallas
Secy.-Treas.—Dorothy Woodley, 230 Roseway Road, El Paso

Greater Dallas Mathematics Association
Pres.—Arthur W. Harris, 4701 Cole Ave.
1st V. Pres.—Frances Freese, 6856 Casa Loma
2nd V. Pres.—Robert W. Fulkerson, 5426 Merrimac
3rd V. Pres.—Lucille Cope, 726 Brookhurst
4th V. Pres.—Elizabeth Cosby, 10215 Plummer
Secy.—Mrs. W. Ogden Kidd, 4121 Greenbrier
Treas.—Mrs. Amy McMurry, 2426 Catherine

Houston Council of Teachers of Mathematics
Pres.—William P. Hand, 1601 Broadway
V. Pres.—Mrs. Velma Dickerson, 1918 Hollister
Secy.—Mrs. Ada Pemberton, 610 W. Main
Treas.—Mrs. Frankie Dunn, 6200 Piping Rock Lane

UTAH

Utah Council of Teachers of Mathematics
Pres.—Dan Guido, 457 Emerson Ave., Salt Lake City
V. Pres.—Anne S. Henriques, University of Utah, Salt Lake City
V. Pres.—Ellis Everett, 143 South 400 West St., St. George
V. Pres.—Daniel D. Coons, 850 East Second St., Salt Lake City
V. Pres.—Alphid Hendrickson, 1150 25th St., Ogden
Secy.-Treas.—Kenneth E. Parsons, 3492 Adams Ave., Ogden

VIRGINIA

Mathematics Section, Virginia Education Association
Pres.—Mrs. E. P. Walker, 320 W. Clifford St., Winchester
V. Pres.—Howard McCue, St. Catherine's School, Richmond
Secy.—Mrs. Owen A. Hoagland, 1035 Woodland Ave., Winchester
Treas.—Norman P. Dodl, Pleasant Hill High School, Shanghai

Richmond Branch of The National Council of Teachers of Mathematics
Pres.—Karl H. Stutzman, 6103 Patterson Ave.
V. Pres.—Florence Tiller, 3124 North Ave.
Secy.—Frances Givens, 2310 First Ave.
Treas.—Peggy Hedrick, 7906 Sycamore Lane

WASHINGTON

The Pudget Sound Council of Teachers of Mathematics
Pres.—Gordon Roberts, Shoreline High School, E. 185th and 1st N.E., Seattle 55
V. Pres.—Ellen Carstairs, Eckstein Junior High School, 3105 E. 75th., Seattle 15
Secy.-Treas.—Zella R. Stewart, Administrative and Service Center, 815 4th Ave. N., Seattle 9

WEST VIRGINIA

West Virginia Council of Mathematics Teachers
Pres.—Jones L. Griffin, 911 Drew St., St. Albans
Pres.-Elect—Mrs. Emma D. Wise, Hinton High School, Hinton
Secy.—Mrs. Joe E. Peck, 1417 Sattes Circle, Nitro
Treas.—Mrs. Carrie B. Campbell, 3609 Bradley Road, Huntington

WISCONSIN

Wisconsin Mathematics Council
Pres.—Ralph J. Cooke, 376 East Thirteenth St., Fond du Lac
V. Pres.—Dr. William Golomski, Marquette University, Milwaukee
V. Pres.—Cassie Raasoch, 413 Chestnut St., Madison
V. Pres.—Joseph Kennedy, Wisconsin High School, Madison
Secy.—Laura Wagner, 634 Short St., Fort Atkinson
Treas.—Roger Sohr, Junior-Senior High School, Waukesha

The following groups have not reported since the date of last printing, April, 1956:

Delaware
Pinellas County Council of Teachers of Mathematics (Florida)
Kentucky Council of Mathematics Teachers
New Mexico
Nassau County Mathematics Teachers Association (New York)
North Carolina
Cleveland Mathematics Club
Mathematics Council of Oklahoma City
Ontario
South Carolina
Tennessee
Utah

Seventeenth Annual Summer Meeting

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Carleton College, Northfield, Minnesota
August 19–21, 1957

Contemporary mathematics, new trends in mathematics education, and current problems of teaching mathematics will be highlighted at the 1957 Summer Meeting of The National Council of Teachers of Mathematics at Carleton College. Outstanding leaders in mathematics and education from all parts of the U.S. will participate in the meetings. Mathematics teachers will share their successful techniques at the many sessions on current teaching problems.

Our accommodations in a rural Minnesota town will be like a retreat at a luxury resort. The facilities at Carleton are noted for their beauty, comfort, and good food. It will be an excellent place to renew friendships and to make new friends. Coffee hours, a picnic, the banquet, and evening recreational periods will provide for your leisure time. Recreational facilities such as golf, tennis, swimming, horseback riding, and trapshooting are available for the athletically inclined. If astronomy is of interest, you may visit the observatory every evening. A delightful hour will be spent on the St. Olaf College Campus at an Open-House coffee hour Tuesday afternoon.

There will be exhibits of the projects, models, or reports of outstanding mathematics students and teachers. The latest in texts and teaching materials will be found in book exhibit and commercial exhibit. Mathematics films and filmstrips will be shown daily with a variety of titles available for preview.

The morning general sessions are indicative of the scope of the program and the outstanding personalities at the meetings. On Monday morning, Dr. William A. Brownell of the University of California will speak on "Dilemma in the Classroom." On Tuesday morning, Dr. John R. Mayor of the American Association for the Advancement of Science will discuss "New Developments in Mathematics Education." The Wednesday morning general session speaker is Dr. Kenneth O. May of Carleton College who will speak on "New Topics for the Mathematics Classroom." These well-known speakers representing elementary, secondary, and college mathematics will discuss the areas emphasized during the meetings, namely, classroom problems, current trends, and contemporary mathematics.

Contemporary mathematics will be presented in a series of seminars on Monday and Tuesday afternoons. These seminars will have lecturers to discuss topics such as Foundations of Arithmetic, Boolean Algebra, Finite Geometry, Statistics, Non-Euclidian Geometry, Number Systems, Game Theory, and the Structure of Mathematics. At these sessions outstanding mathematics teachers will present the elementary, basic ideas of these topics with ample time allowed for participation and questions from the audience.

Monday morning sectional meetings at each level of instruction will have speakers discussing the problems in the elementary school, junior high school, senior high school, integrated mathematics, and teacher education. Laboratory sections on the elementary, junior high school, and senior high school levels will be held late Monday and Tuesday afternoons.

The major classroom problems of the mathematics teacher will be emphasized in the Tuesday morning sectional meetings. These meetings will be concerned with the curriculum, materials of instruction, remedial instruction, programs for the gifted students, and evaluation. These topics will be discussed by mathematics

teachers who will describe their successful classroom experiences. These sections break away from grade level organization and present each topic over a broad range of grade levels.

The Wednesday morning sectional meetings will discuss how to teach the big ideas of mathematics. Classroom teachers will present concrete, practical suggestions that they have used in their classrooms. The big ideas being covered are arithmetical processes, problem solving, the function concept, methods of proof, statistics, and measurement.

The program for the summer meeting has been based on the suggestions of many classroom teachers. It has been planned to give you the information, inspiration, and materials you need to become more successful in teaching mathematics. Don't miss this opportunity for an inexpensive, delightful short course.

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of the MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D. C.

NCTM convention dates

JOINT MEETING WITH NEA AND NSTA

July 1, 1957
Philadelphia, Pennsylvania
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D. C.

SUMMER MEETING

August 19–21, 1957
Carleton College, Northfield, Minnesota
Margaret Linster, St. Louis Park High School, Minneapolis 16, Minnesota, or Kenneth O. May, Carleton College, Northfield, Minnesota

ANNUAL MEETING

April 9–12, 1958
Hotel Cleveland, Cleveland, Ohio
Lawrence Hyman, Board of Education, Cleveland, Ohio

Other professional dates

Eighth Annual Mathematics Institute of Louisiana State University

June 23–29, 1957
Louisiana State University, Baton Rouge, Louisiana
Houston T. Karnes, Louisiana State University, Baton Rouge, Lousiana

Summer Session for Mathematics Teachers

July 1–August 9, 1957
Syracuse University, Syracuse 10, New York
Robert B. Davis, Department of Mathematics, Syracuse University, Syracuse 10, New York

Shell Merit Fellowship Program for Mathematics and Science Teachers

July 1–August 11, 1957
Cornell University, Ithaca, New York
Phillip G. Johnson, Cornell University, Ithaca, New York

Arithmetic Workshop

August 12–16, 1957
University of Minnesota, Minneapolis, Minnesota
Arden K. Ruddell, College of Education, University of Minnesota, Minneapolis 14, Minnesota

Ninth Annual Institute of the Association of Teachers of Mathematics in New England

August 21–28, 1957
Dartmouth College, Hanover, New Hampshire
Alma A. Sargent, 125 N. State Street, Concord, New Hampshire

Annual Mathematics Institute of the Florida Council of Teachers of Mathematics

August 22–23, 1957
St. Petersburg Junior College, St. Petersburg, Florida
Kenneth P. Kidd, University of Florida, Gainesville, Florida

Registrations at Annual Joint Meeting with the NEA

The National Council of Teachers of Mathematics, Portland, Oregon, July 2, 1956

Alabama	1
Arizona	1
California	6
Colorado	1
Connecticut	1
District of Columbia	1
Illinois	3
Indiana	3
Iowa	1
Kansas	2
Kentucky	1
Maine	1
Maryland	8
Michigan	7
Minnesota	1
Missouri	3
New Mexico	3
North Carolina	1
Ohio	2
Oklahoma	3
Oregon	71
Pennsylvania	3
South Dakota	1
Tennessee	2
Texas	3
Utah	2
Washington	6
Wisconsin	1
Alaska	1
Hawaii	1
Foreign	1
Total	142

Registrations at the Sixteenth Summer Meeting

The National Council of Teachers of Mathematics, Los Angeles, California, August 19–22, 1956

Arkansas	1
Arizona	4
California	269
Colorado	2
Connecticut	1
District of Columbia	2
Illinois	9
Indiana	4
Iowa	1
Kentucky	1
Louisiana	3
Maryland	1
Michigan	3
Minnesota	4
Missouri	3
Nebraska	1
Nevada	1
New Jersey	2
New Mexico	4
New York	5
Ohio	3
Oklahoma	2
Oregon	7
Tennessee	2
Texas	7
Utah	1
Virginia	1
Washington	1
Wisconsin	1
Canada	2
Foreign	1
Total	349

Fields of interest of the registrants as indicated by the check marks on the registration cards:

Senior High School	151
Junior High School	81
Elementary	47
College	27
Teacher Training	26
Junior College	20
Supervision	18
Other	4

Number of registrants who stated that they are members of the Council **221**

Number of registrants who stated that they are members of an affiliated group of the Council **88**

Notice

The Committee on Nominations and Elections is anxious to have suggestions from National Council members for candidates for the 1958 election. Officers to be elected will be as follows: president, vice-president for high school, vice-president for elementary, and three directors.

Suggestions should be given to any member of the Committee (see November, 1956, issue of THE MATHEMATICS TEACHER) or should be mailed to the Chairman, Clifford Bell, Department of Mathematics, University of California, Los Angeles 24, California, not later than May 28, 1957.

Technician manpower shortage

The *New York Herald Tribune*, December 30, 1956, carried a 32-page "1957 Preview of Technical Positions, a Directory of Career Opportunities for Engineers, Scientists and Technicians." Industries from coast to coast advertised their needs for technical help for a total of 27 of these pages.

The *Tribune's* Preview dramatizes in its very volume today's unique demand for manpower in the technical area, and educators are well aware of the shortage of engineers and scientists. But note that the Preview title also includes *technicians* in addition to engineers and scientists.

Technician is a word much less familiar to the education profession than is engineer or scientist. In fact, the word technician seldom connotes similar meanings within a group of educators or business men. The Technical Institute Division of the American Society for Engineering Education, in its publications, describes the technician occupations as those "between the trades and the professional engineer."

Emphasis on technician training courses is on theory and technology—particularly on basic mathematics (algebra through applied trigonometry), science, and elementary engineering courses—rather than on the manipulative emphasis of trade training. Some typical job opportunities for technicians are: drafting, engineer's aide, laboratory assistant, inspection, technical sales, electronics technician, refrigeration technician, mechanical technician, and mid-level supervision.

This writer has studied technician manpower needs in the San Francisco Bay area during the past three years. The study includes 30 organizations employing 57,000 people. The study, done by intensive personal interviews, indicates a need proportionate to the *Herald Tribune's* voluminous Preview. The study further reveals that an increasing number of industries, private and civil-service, are establishing a definitive structure of technician positions to attract career people; that demand for workers with technician training is rapidly increasing with the advances of automation; that supply of technicians in many specialties is short.

Industry, the secondary schools, and colleges have a mutual interest in working together to define training needs, occupational guidance, and career opportunities for technicians. The personal-interview study method is perhaps the best way for a school to initiate such action on a community basis.—*Kenneth C. Skeen, Concord, California.*